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On Leonardo, Leonardo–Lucas and modified Leonardo elliptic quaternions and their matrix representations

Abstract. In this paper, we present a new class of elliptic quaternions that incorporate Leonardo, Leonardo–Lucas and modified Leonardo numbers into their components. We explore some fundamental properties associated with these numbers. In particular, we obtain recurrence relations, generating function, Binet formula of these sequences and by using Binet formula we derive Vajda, Cassini, Catalan and d'Ocagne identities. Lastly, we investigate two different matrix representations of these numbers.

1. Introduction. Recently, Catarino and Borges [3] explored the recurrence relations and various properties of Leonardo numbers. Following their work, Alp and Koçer [1] investigated additional interesting properties of these numbers. Kuhapatanakul and Juthamas [8] extended the study to generalized Leonardo numbers, examining their matrix representation. Additionally, Karatas [7] defined complex Leonardo numbers and analyzed their combinatorial properties. İşbilir et al. [6] investigated Pauli–Leonardo quaternions.

Further developments on Leonardo numbers, their generalizations, and interesting properties can be found in [2, 9, 12, 11, 14, 16, 17].

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Özdemir [10] explained how, for a given ellipsoid, one can define a suitable scalar product and vector product. He introduced elliptical inner product, elliptical vector product and he aimed to describe the motion on an ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ as a rotation using the proper inner product, vector and elliptical orthogonal matrices introduced in that elliptical scalar product space. He also defined the elliptic quaternions and showed a relation between elliptical rotation and elliptic quaternions.

Let $\{e_0, e_1, e_2, e_3\}$ be four basic elements satisfying the equalities $e_0 = 1$, $e_1^2 = -\alpha$, $e_2^2 = -\beta$, $e_3^2 = -\gamma$ and

$$(1.1) \quad e_1e_2 = \frac{\Delta}{\gamma}e_3 = -e_2e_1,\ e_2e_3 = \frac{\Delta}{\alpha}e_1 = -e_3e_2,\ e_3e_1 = \frac{\Delta}{\beta}e_2 = -e_1e_3$$

where $\alpha, \beta, \gamma \in \mathbb{R}^+$ and $\Delta = \sqrt{\alpha\beta\gamma}$. The set of elliptic quaternions is defined by the formula

$$(1.2) \qquad \mathbb{H}_{\alpha,\beta,\gamma} = \{Q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3 :\ q_0, q_1, q_2, q_3 \in \mathbb{R}\}.$$

This set is an associative, noncommutative division algebra with the basic elements $\{e_0, e_1, e_2, e_3\}$ and it is a 4-dimensional algebra over real numbers. If α, β and γ are equal to 1, we have the real quaternion algebra. Thus, elliptic quaternions are an extension of real quaternions with the elliptic inner and vector product. Furthermore, the following table shows the products of dual elliptic quaternion units.

$\cdot$	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	$-\alpha$	$(\Delta/\gamma)\, e_3$	$(-\Delta/\beta)\, e_2$
e_2	e_2	$(-\Delta/\gamma)\, e_3$	$-\beta$	$(\Delta/\alpha)\, e_1$
e_3	e_3	$(\Delta/\beta)\, e_2$	$(-\Delta/\alpha)\, e_1$	$-\gamma$

Table 1. The products of elliptic quaternionic units.

Additionally, as in the case of quaternions, it is possible to define right and left multiplications of two elliptic quaterions using 4×4 real matrices. For an elliptic quaternion $Q = q_0 + q_1e_1 + q_2e_2 + q_3e_3$, consider the transformations

$$\begin{aligned} \varphi : \mathbb{H}_{\alpha,\beta,\gamma} &\longrightarrow \mathbb{H}_{\alpha,\beta,\gamma} \\ P &\longrightarrow \varphi(P) = QP \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mu : \mathbb{H}_{\alpha,\beta,\gamma} &\longrightarrow \mathbb{H}_{\alpha,\beta,\gamma} \\ P &\longrightarrow \mu(P) = PQ. \end{aligned}$$

The following 4×4 matrices corresponding to these transformations, respectively, provide the left and right product matrices of Q:

$$\mathscr{L}_Q=\begin{bmatrix} q_0 & -\alpha q_1 & -\beta q_2 & -\gamma q_3\\ q_1 & q_0 & -\frac{\Delta}{\alpha}q_3 & \frac{\Delta}{\alpha}q_2\\ q_2 & \frac{\Delta}{\beta}q_3 & q_0 & -\frac{\Delta}{\beta}q_1\\ q_3 & -\frac{\Delta}{\gamma}q_2 & \frac{\Delta}{\gamma}q_1 & q_0\end{bmatrix},\quad \mathscr{R}_Q=\begin{bmatrix} q_0 & -\alpha q_1 & -\beta q_2 & -\gamma q_3\\ q_1 & q_0 & \frac{\Delta}{\alpha}q_3 & -\frac{\Delta}{\alpha}q_2\\ q_2 & -\frac{\Delta}{\beta}q_3 & q_0 & \frac{\Delta}{\beta}q_1\\ q_3 & \frac{\Delta}{\gamma}q_2 & -\frac{\Delta}{\gamma}q_1 & q_0\end{bmatrix}.$$

For detailed information about these concepts, we refer to reader to [4, 10, 13].

2. Basic concepts and notions. This section contains some definitions and theorems that will be used in the following sections of the paper.

For $n\geq 2$, the sequence of Fibonacci numbers $\{\mathrm{F}_n\}$ is defined by

$$\mathrm{F}_n=\mathrm{F}_{n-1}+\mathrm{F}_{n-2},\quad \mathrm{F}_0=0,\ \mathrm{F}_1=1 \tag{2.1}$$

and the sequence of Lucas numbers $\{\mathrm{L}_n\}$ is defined by

$$\mathrm{L}_n=\mathrm{L}_{n-1}+\mathrm{L}_{n-2},\quad \mathrm{L}_0=2,\ \mathrm{L}_1=1. \tag{2.2}$$

Let ψ and ϕ be the roots of the characteristic equation $t^2-t-1=0$ corresponding to formulas (2.1)–(2.2). Solving the characteristic equation, we obtain the distinct roots $\psi=\frac{1+\sqrt{5}}{2}$ and $\phi=\frac{1-\sqrt{5}}{2}$. Then the Binet formulas for Fibonacci (F_n) and Lucas (L_n) numbers are:

$$\mathrm{F}_n=\frac{\psi^n-\phi^n}{\psi-\phi}\quad\text{and}\quad \mathrm{L}_n=\psi^n+\phi^n.$$

Note that we have the following equations:

$$2\mathrm{F}_{n+1}=\mathrm{F}_n+\mathrm{L}_n,\quad \mathrm{L}_n=\mathrm{F}_{n-1}+\mathrm{F}_{n+1},\quad \mathrm{L}_n=\mathrm{F}_{n+2}-\mathrm{F}_{n-2}. \tag{2.3}$$

Similar to the Fibonacci and Lucas numbers, the Leonardo numbers, which were first introduced by Dijkstra [5] in 1981 in a sorting algorithm, satisfy the recurrence relation

$$\mathcal{L}_n=\mathcal{L}_{n-1}+\mathcal{L}_{n-2}+1,\quad n\geq 2 \tag{2.4}$$

with the initial conditions $\mathcal{L}_0=\mathcal{L}_1=1$. This sequence can be seen in OEIS [A001595]. Furthermore, there are two number sequences called Leonardo–Lucas and modified Leonardo introduced by Soykan in [14]. For $n\geq 2$, they are defined as

$$\mathcal{L}_{\mathrm{L},n}=\mathcal{L}_{\mathrm{L},n-1}+\mathcal{L}_{\mathrm{L},n-2}-1,\quad \mathcal{L}_{\mathrm{L},0}=3,\ \mathcal{L}_{\mathrm{L},1}=2 \tag{2.5}$$

and

$$\mathcal{M}_n=\mathcal{M}_{n-1}+\mathcal{M}_{n-2}+1,\quad \mathcal{M}_0=0,\ \mathcal{M}_1=1, \tag{2.6}$$

respectively. These sequences can be seen in OEIS [A001612] and [A000071], respectively. The sequences $\{\mathcal{L}_n\}$, $\{\mathcal{L}_{\mathrm{L},n}\}$ and $\{\mathcal{M}_n\}$ satisfy the third order linear recurrences as follows:

$$\begin{aligned} &\mathcal{L}_n = 2\mathcal{L}_{n-1} - \mathcal{L}_{n-3}, \quad \mathcal{L}_{\mathrm{L},n} = 2\mathcal{L}_{\mathrm{L},n-1} - \mathcal{L}_{\mathrm{L},n-3}, \\ &\mathcal{M}_n = 2\mathcal{M}_{n-1} - \mathcal{M}_{n-3}. \end{aligned} \tag{2.7}$$

Let ψ, ϕ and ω be roots of the characteristic equation $t^3 - 2t^2 + 1 = \left(t^2 - t - 1\right)(t-1) = 0$ corresponding to equations (2.4)–(2.6). By solving the equation, we get three distinct roots as follows:

$$\psi = \frac{1+\sqrt{5}}{2}, \quad \phi = \frac{1-\sqrt{5}}{2} \quad \text{and} \quad \omega = 1. \tag{2.8}$$

Note that

$$\begin{aligned} \psi + \phi + \omega = 2 \quad &\text{or} \quad \psi + \phi = 1,\ \psi - \phi = \sqrt{5} \\ \psi\phi + \psi\omega + \phi\omega = 0 \\ \psi\phi\omega = -1 \quad &\text{or} \quad \psi\phi = -1 \end{aligned} \tag{2.9}$$

and moreover $\psi^2 + 1 = \sqrt{5}\psi$, $\phi^2 + 1 = -\sqrt{5}\phi$, $\psi^2 + \phi^2 = 3, \psi^2 - \phi^2 = \sqrt{5}$ and $\psi^3 + \phi^3 = 4$. On the other hand,

$$\mathcal{L}_n = \frac{2\left(\psi^{n+1} - \phi^{n+1}\right)}{\psi - \phi} - 1, \tag{2.10}$$

$$\mathcal{L}_{\mathrm{L},n} = \psi^n + \phi^n + 1, \tag{2.11}$$

$$\mathcal{M}_n = \frac{\psi^{n+2} - \phi^{n+2}}{\psi - \phi} - 1 \tag{2.12}$$

are the Binet formulas of Leonardo, Leonardo–Lucas and modified Leonardo numbers, respectively. From the Binet formulas of Fibonacci and Lucas numbers, we get

$$\mathcal{L}_n = 2\mathrm{F}_{n+1} - 1, \quad \mathcal{L}_{\mathrm{L},n} = \mathrm{L}_n + 1, \quad \mathcal{M}_n = \mathrm{F}_{n+2} - 1. \tag{2.13}$$

In addition, we have the following interrelations:

$$\begin{aligned} 5\mathcal{L}_n &= 2\mathrm{L}_{n+1} + 4\mathrm{L}_n - 5, \quad \mathcal{L}_{\mathrm{L},n} = 2\mathrm{F}_{n+1} - \mathrm{F}_n + 1, \\ 5\mathcal{M}_n &= 3\mathrm{L}_{n+1} + \mathrm{L}_n - 5. \end{aligned} \tag{2.14}$$

Furthermore, there are the following relations among Leonardo, Leonardo-Lucas and modified Leonardo numbers:

$$\begin{aligned} 2\mathcal{L}_{\mathrm{L},n} &= \mathcal{L}_{n-2} + \mathcal{L}_n + 2, \\ 2\mathcal{M}_n &= \mathcal{L}_{n+1} - 1, \\ \mathcal{M}_n &= \mathcal{L}_{\mathrm{L},n} + \mathcal{L}_{\mathrm{L},n-1} - \mathrm{F}_n - 3. \end{aligned} \tag{2.15}$$

In [1], the $(-n)$th Leonardo number with negative subscript is defined as follows:

$$\mathcal{L}_{-n} = (-1)^{-1}\left(\mathcal{L}_{n-2} + 1\right) - 1, \quad \text{for } n \geq 2. \tag{2.16}$$

Additionally, the following recurrence relation for negative subscripted Leonardo numbers is presented in [17]:

$$\mathcal{L}_{-n} = -\mathcal{L}_{-n+1} + \mathcal{L}_{-n+2} - 1, \quad \text{for } n \geq 0 \tag{2.17}$$

and some values of negative subscripted Leonardo numbers are $\mathcal{L}_{-1} = -1$, $\mathcal{L}_{-2} = 1$, $\mathcal{L}_{-3} = -3$, $\mathcal{L}_{-4} = 3$ and $\mathcal{L}_{-5} = -7$.

Moreover, the matrix representation of Leonardo numbers was presented in [1] with the following 3×3 matrix:

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \tag{2.18}$$

and some properties of these numbers were studied with respect to the matrix.

Now we can introduce the Fibonacci and Lucas elliptic quaternions. These were investigated by Tan et al. in [15] as follows.

The nth Fibonacci and Lucas elliptic quaternions are defined respectively by

$$\begin{aligned} \mathcal{Q}\mathrm{F}_n &= \mathrm{F}_n + \mathrm{F}_{n+1}e_1 + \mathrm{F}_{n+2}e_2 + \mathrm{F}_{n+3}e_3, \\ \mathcal{Q}\mathrm{L}_n &= \mathrm{L}_n + \mathrm{L}_{n+1}e_1 + \mathrm{L}_{n+2}e_2 + \mathrm{L}_{n+3}e_3, \end{aligned} \tag{2.19}$$

where F_n and L_n are the nth Fibonacci and Lucas numbers, respectively. Here e_1, e_2 and e_3 are the basis and satisfy the relation in (1.1).

Furthermore, the Fibonacci and Lucas elliptic quaternions satisfy the following recurrence relations:

$$\begin{aligned} \mathcal{Q}\mathrm{F}_n &= \mathcal{Q}\mathrm{F}_{n-1} + \mathcal{Q}\mathrm{F}_{n-2}, \quad n \geq 2, \\ \mathcal{Q}\mathrm{L}_n &= \mathcal{Q}\mathrm{L}_{n-1} + \mathcal{Q}\mathrm{L}_{n-2}, \quad n \geq 2, \end{aligned} \tag{2.20}$$

respectively, with the initial conditions $\mathcal{Q}\mathrm{F}_0 = e_1 + e_2 + 2e_3$, $\mathcal{Q}\mathrm{F}_1 = 1 + e_1 + 2e_2 + 3e_3$ and $\mathcal{Q}\mathrm{L}_0 = 2 + e_1 + 3e_2 + 4e_3$, $\mathcal{Q}\mathrm{L}_1 = 1 + 3e_1 + 4e_2 + 7e_3$.

3. Leonardo, Leonardo–Lucas and modified Leonardo elliptic quaternions. In this section, we construct Leonardo, Leonardo–Lucas and modified Leonardo elliptic quaternions and establish some properties of these number sequences. Moreover, we give matrix representations of them.

Definition 3.1. For $n \geq 1$, the nth Leonardo, Leonardo–Lucas and modified Leonardo elliptic quaternions (briefly, LEQ, LLEQ, MLEQ) are defined by

$$\begin{aligned} \mathcal{Q}\mathcal{L}_n &= \mathcal{L}_n + \mathcal{L}_{n+1}e_1 + \mathcal{L}_{n+2}e_2 + \mathcal{L}_{n+3}e_3, \\ \mathcal{Q}\mathcal{L}_{\mathrm{L},n} &= \mathcal{L}_{\mathrm{L},n} + \mathcal{L}_{\mathrm{L},n+1}e_1 + \mathcal{L}_{\mathrm{L},n+2}e_2 + \mathcal{L}_{\mathrm{L},n+3}e_3, \\ \mathcal{Q}\mathcal{M}_n &= \mathcal{M}_n + \mathcal{M}_{n+1}e_1 + \mathcal{M}_{n+2}e_2 + \mathcal{M}_{n+3}e_3, \end{aligned} \tag{3.1}$$

respectively, where e_1, e_2 and e_3 are elliptic quaternionic units and $\mathcal{L}_n$, $\mathcal{L}_{\mathrm{L},n}$ and $\mathcal{M}_n$ are the nth Leonardo, Leonardo–Lucas and modified Leonardo numbers, respectively. The set of all LEQ, LLEQ and MLEQ are denoted by $\mathbb{LH}$, $\mathbb{LH}_{\mathrm{L}}$ and $\mathbb{MH}$, respectively.

The sequences $\{\mathcal{QL}_n\}$, $\{\mathcal{QL}_{\mathrm{L},n}\}$ and $\{\mathcal{QM}_n\}$ satisfy the following third-order linear recurrence relations, respectively:

$$\mathcal{QL}_n = 2\mathcal{QL}_{n-1} - \mathcal{QL}_{n-3}, \tag{3.2}$$

$$\mathcal{QL}_{\mathrm{L},n} = 2\mathcal{QL}_{\mathrm{L},n-1} - \mathcal{QL}_{\mathrm{L},n-3}, \tag{3.3}$$

$$\mathcal{QM}_n = 2\mathcal{QM}_{n-1} - \mathcal{QM}_{n-3}. \tag{3.4}$$

Theorem 3.2. *For $n \geq 2$, LEQ, LLEQ and MLEQ satisfy the recurrence relations*

$$\mathcal{QL}_n = \mathcal{QL}_{n-1} + \mathcal{QL}_{n-2} + Q^*, \tag{3.5}$$

$$\mathcal{QL}_{\mathrm{L},n} = \mathcal{QL}_{\mathrm{L},n-1} + \mathcal{QL}_{\mathrm{L},n-2} - Q^*, \tag{3.6}$$

$$\mathcal{QM}_n = \mathcal{QM}_{n-1} + \mathcal{QM}_{n-2} + Q^* \tag{3.7}$$

with the initial conditions

$$\begin{aligned} \mathcal{QL}_0 &= 1 + e_1 + 3e_2 + 5e_3, & \mathcal{QL}_1 &= 1 + 3e_1 + 5e_2 + 9e_3, \\ \mathcal{QL}_{\mathrm{L},0} &= 3 + 2e_1 + 4e_2 + 5e_3, & \mathcal{QL}_{\mathrm{L},1} &= 2 + 4e_1 + 5e_2 + 8e_3, \\ \mathcal{QM}_0 &= e_1 + 2e_2 + 4e_3, & \mathcal{QM}_1 &= 1 + 2e_1 + 4e_2 + 7e_3, \end{aligned}$$

where $Q^* = 1 + e_1 + e_2 + e_3$.

Proof. It can be easily proved by using the equations (2.4)–(2.6) and (3.1). □

Theorem 3.3. *The generating functions for LEQ, LLEQ and MLEQ are*

$$\begin{aligned} \mathrm{GF}_{\mathcal{QL}_n}(t) &= \frac{\mathcal{QL}_0 + (\mathcal{QL}_1 - 2\mathcal{QL}_0)\,t + (\mathcal{QL}_2 - 2\mathcal{QL}_1)\,t^2}{t^3 - 2t + 1}, \\ \mathrm{GF}_{\mathcal{QL}_{\mathrm{L},n}}(t) &= \frac{\mathcal{QL}_{\mathrm{L},0} + (\mathcal{QL}_{\mathrm{L},1} - 2\mathcal{QL}_{\mathrm{L},0})\,t + (\mathcal{QL}_{\mathrm{L},2} - 2\mathcal{QL}_{\mathrm{L},1})\,t^2}{t^3 - 2t + 1}, \\ \mathrm{GF}_{\mathcal{QM}_n}(t) &= \frac{\mathcal{QM}_0 + (\mathcal{QM}_1 - 2\mathcal{QM}_0)\,t + (\mathcal{QM}_2 - 2\mathcal{QM}_1)\,t^2}{t^3 - 2t + 1}. \end{aligned} \tag{3.8}$$

Proof. Let $\mathrm{GF}_{\mathcal{QL}_n}(t)$ be the generating function for LEQ such that

$$\mathrm{GF}_{\mathcal{QL}_n}(t) = \mathcal{QL}_0 + \mathcal{QL}_1 t + \mathcal{QL}_2 t^2 + \cdots + \mathcal{QL}_n t^n + \cdots. \tag{3.9}$$

Multiplying both sides of (3.9) by $-2t$ and t^3, we have

$$\begin{aligned} -2t\mathrm{GF}_{\mathcal{QL}_n}(t) &= -2\mathcal{QL}_0 t - 2\mathcal{QL}_1 t^2 - 2\mathcal{QL}_2 t^3 - \cdots - 2\mathcal{QL}_n t^{n+1} + \cdots \\ t^3\mathrm{GF}_{\mathcal{QL}_n}(t) &= \mathcal{QL}_0 t^3 + \mathcal{QL}_1 t^4 + \mathcal{QL}_2 t^5 + \cdots + \mathcal{QL}_n t^{n+3} + \cdots. \end{aligned}$$

We have anticipated result (3.8) by using the initial conditions of LEQ and making the necessary operations. Other equations can be proved in the same way. □

Corollary 3.4. *Considering* (2.13) *and* (2.20), *we obtain*

$$\begin{aligned}\mathcal{QL}_n &= \mathcal{L}_n + \mathcal{L}_{n+1}e_1 + \mathcal{L}_{n+2}e_2 + \mathcal{L}_{n+3}e_3 \\ &= 2\mathrm{F}_{n+1} - 1 + (2\mathrm{F}_{n+2} - 1)\,e_1 + (2\mathrm{F}_{n+3} - 1)\,e_2 + (2\mathrm{F}_{n+4} - 1)\,e_3 \\ &= 2\mathcal{Q}\mathrm{F}_{n+1} - Q^*,\end{aligned} \tag{3.10}$$

$$\begin{aligned}\mathcal{QL}_{\mathrm{L},n} &= \mathcal{L}_{\mathrm{L},n} + \mathcal{L}_{\mathrm{L},n+1}e_1 + \mathcal{L}_{\mathrm{L},n+2}e_2 + \mathcal{L}_{\mathrm{L},n+3}e_3 \\ &= \mathrm{L}_n + 1 + (\mathrm{L}_{n+1} + 1)\,e_1 + (\mathrm{L}_{n+2} + 1)\,e_2 + (\mathrm{L}_{n+3} + 1)\,e_3 \\ &= \mathcal{Q}\mathrm{L}_n + Q^*\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\mathcal{QM}_n &= \mathcal{M}_n + \mathcal{M}_{n+1}e_1 + \mathcal{M}_{n+2}e_2 + \mathcal{M}_{n+3}e_3 \\ &= \mathrm{F}_{n+2} - 1 + (\mathrm{F}_{n+3} - 1)\,e_1 + (\mathrm{F}_{n+4} - 1)\,e_2 + (\mathrm{F}_{n+5} - 1)\,e_3 \\ &= \mathcal{Q}\mathrm{F}_{n+2} - Q^*\end{aligned} \tag{3.12}$$

If we consider (2.3) *and* (3.10)–(3.12), *we have*

$$\begin{aligned}\mathcal{QL}_n &= 2\mathrm{F}_{n+1} - 1 + (2\mathrm{F}_{n+2} - 1)\,e_1 + (2\mathrm{F}_{n+3} - 1)\,e_2 + (2\mathrm{F}_{n+4} - 1)\,e_3 \\ &= \mathrm{F}_n + \mathrm{L}_n + (\mathrm{F}_{n+1} + \mathrm{L}_{n+1})\,e_1 + (\mathrm{F}_{n+2} + \mathrm{L}_{n+2})\,e_2 \\ &\quad + (\mathrm{F}_{n+3} + \mathrm{L}_{n+3})\,e_3 - Q^* \\ &= \mathcal{Q}\mathrm{F}_n + \mathcal{Q}\mathrm{L}_n - Q^*,\end{aligned}$$

$$\begin{aligned}\mathcal{QL}_{\mathrm{L},n} &= \mathrm{L}_n + 1 + (\mathrm{L}_{n+1} + 1)\,e_1 + (\mathrm{L}_{n+2} + 1)\,e_2 + (\mathrm{L}_{n+3} + 1)\,e_3 \\ &= \mathrm{F}_{n-1} + \mathrm{F}_{n+1} + (\mathrm{F}_n + \mathrm{F}_{n+2})\,e_1 + (\mathrm{F}_{n+1} + \mathrm{F}_{n+3})\,e_2 \\ &\quad + (\mathrm{F}_{n+2} + \mathrm{F}_{n+4})\,e_3 + Q^* \\ &= \mathcal{Q}\mathrm{F}_{n-1} + \mathcal{Q}\mathrm{F}_{n+1} + Q^*\end{aligned}$$

and

$$\begin{aligned}\mathcal{QM}_n &= \mathrm{L}_n + \mathrm{F}_{n-2} - 1 + (\mathrm{L}_{n+1} + \mathrm{F}_{n-1} - 1)\,e_1 + (\mathrm{L}_{n+2} + \mathrm{F}_n - 1)\,e_2 \\ &\quad + (\mathrm{L}_{n+3} + \mathrm{F}_{n+1} - 1)\,e_3 \\ &= \mathcal{Q}\mathrm{L}_n + \mathcal{Q}\mathrm{F}_{n-2} - Q^*\end{aligned}$$

where $\mathcal{Q}\mathrm{F}_n$ *and* $\mathcal{Q}\mathrm{L}_n$ *are the nth Fibonacci and Lucas elliptic quaternions, respectively.*

Corollary 3.5. *Considering* (2.14) *and* (2.20)*, we obtain*

$$\begin{aligned}5\mathcal{QL}_n &= 5\left(\mathcal{L}_n + \mathcal{L}_{n+1}e_1 + \mathcal{L}_{n+2}e_2 + \mathcal{L}_{n+3}e_3\right)\\ &= 2\mathrm{L}_{n+1} + 4\mathrm{L}_n - 5 + \left(2\mathrm{L}_{n+2} + 4\mathrm{L}_{n+1} - 5\right)e_1\\ &\quad + \left(2\mathrm{L}_{n+3} + 4\mathrm{L}_{n+2} - 5\right)e_2 + \left(2\mathrm{L}_{n+4} + 4\mathrm{L}_{n+3} - 5\right)e_3\\ &= 2\mathcal{Q}\mathrm{L}_{n+1} + 4\mathcal{Q}\mathrm{L}_n - 5Q^*,\end{aligned}$$

$$\begin{aligned}\mathcal{QL}_n &= \mathcal{L}_{\mathrm{L},n} + \mathcal{L}_{\mathrm{L},n+1}e_1 + \mathcal{L}_{\mathrm{L},n+2}e_2 + \mathcal{L}_{\mathrm{L},n+3}e_3\\ &= 2\mathrm{F}_{n+1} - \mathrm{F}_n + 1 + \left(2\mathrm{F}_{n+2} - \mathrm{F}_{n+1} + 1\right)e_1\\ &\quad + \left(2\mathrm{F}_{n+3} - \mathrm{F}_{n+2} + 1\right)e_2 + \left(2\mathrm{F}_{n+4} - \mathrm{F}_{n+3} + 1\right)e_3\\ &= 2\mathcal{Q}\mathrm{F}_{n+1} - \mathcal{Q}\mathrm{F}_n + Q^*\end{aligned}$$

and

$$\begin{aligned}5\mathcal{QM}_n &= 5\left(\mathcal{L}_{\mathrm{L},n} + \mathcal{L}_{\mathrm{L},n+1}e_1 + \mathcal{L}_{\mathrm{L},n+2}e_2 + \mathcal{L}_{\mathrm{L},n+3}e_3\right)\\ &= 5\mathrm{L}_{n+1} + \mathrm{L}_n - 5 + \left(3\mathrm{L}_{n+2} + \mathrm{L}_{n+1} - 5\right)e_1\\ &\quad + \left(3\mathrm{L}_{n+3} + \mathrm{L}_{n+2} - 5\right)e_2 + \left(3\mathrm{L}_{n+4} + \mathrm{L}_{n+3} - 5\right)e_3\\ &= 3\mathcal{Q}\mathrm{L}_{n+1} + \mathcal{Q}\mathrm{L}_n - 5Q^*,\end{aligned}$$

where $\mathcal{Q}\mathrm{F}_n$ *and* $\mathcal{Q}\mathrm{L}_n$ *are the nth Fibonacci and Lucas elliptic quaternions, respectively.*

Corollary 3.6. *By using the equation* (2.16)*, we can also define the negative subscripted LEQ as follows:*

$$\mathcal{QL}_{-n} = \mathcal{L}_{-n} + \mathcal{L}_{-n+1}e_1 + \mathcal{L}_{-n+2}e_2 + \mathcal{L}_{-n+3}e_3, \quad \text{for } n > 0. \tag{3.13}$$

Theorem 3.7. *Let* $\mathcal{QL}_{-n}$ *be the* $(-n)$*th LEQ, then*

$$\begin{aligned}\mathcal{QL}_{-n} &= (-1)^n\left[\left(\mathcal{L}_{n-2}+1\right) + \left(\mathcal{L}_{n-4}+1\right)e_2\right]\\ &\quad + (-1)^{n-1}\left[\left(\mathcal{L}_{n-3}+1\right)e_1 + \left(\mathcal{L}_{n-5}+1\right)e_3\right] - Q^*, \quad \text{for } n \geq 2,\\ \mathcal{QL}_{-n} &= -\mathcal{QL}_{-n+1} + \mathcal{QL}_{-n+2} - Q^*, \quad \text{for } n > 0.\end{aligned}$$

Proof. It can be easily proved by using the equations (2.16) and (2.17). □

Theorem 3.8. *The Binet's formulas of LEQ, LLEQ and MLEQ are*

$$\mathcal{QL}_n = \frac{2\psi^{n+1}\psi^* - 2\phi^{n+1}\phi^*}{\psi - \phi} - Q^*, \tag{3.14}$$

$$\mathcal{QL}_{\mathrm{L},n} = \psi^n\psi^* + \phi^n\phi^* + Q^*, \tag{3.15}$$

$$\mathcal{QM}_n = \frac{\psi^{n+2}\psi^* - \phi^{n+2}\phi^*}{\psi - \phi} - Q^* \tag{3.16}$$

where $\psi^* = 1 + \psi e_1 + \psi^2 e_2 + \psi^3 e_3$ *and* $\phi^* = 1 + \phi e_1 + \phi^2 e_2 + \phi^3 e_3$.

Proof. By (2.10)–(2.12) and (3.1), we obtain

$$\begin{aligned}
\mathcal{QL}_n &= \mathcal{L}_n + \mathcal{L}_{n+1}e_1 + \mathcal{L}_{n+2}e_2 + \mathcal{L}_{n+3}e_3 \\
&= \frac{2\left(\psi^{n+1}-\phi^{n+1}\right)}{\psi-\phi} - 1 + \left(\frac{2\left(\psi^{n+1}-\phi^{n+1}\right)}{\psi-\phi} - 1\right)e_1 \\
&\quad + \left(\frac{2\left(\psi^{n+2}-\phi^{n+2}\right)}{\psi-\phi} - 1\right)e_2 + \left(\frac{2\left(\psi^{n+3}-\phi^{n+3}\right)}{\psi-\phi} - 1\right)e_3 \\
&= \frac{2\psi^{n+1}}{\psi-\phi}\left(1+\psi e_1+\psi^2 e_2+\psi^3 e_3\right) - \frac{2\phi^{n+1}}{\psi-\phi}\left(1+\phi e_1+\phi^2 e_2+\phi^3 e_3\right) \\
&\quad - \frac{\psi}{\psi-\phi}\left(1+e_1+e_2+e_3\right) + \frac{\phi}{\psi-\phi}\left(1+e_1+e_2+e_3\right) \\
&= \frac{2\psi^{n+1}\psi^* - 2\phi^{n+1}\phi^*}{\psi-\phi} - Q^*.
\end{aligned}$$

By (2.11) and (3.1), we get

$$\begin{aligned}
\mathcal{QL}_{\mathrm{L},n} &= \mathcal{L}_{\mathrm{L},n} + \mathcal{L}_{\mathrm{L},n+1}e_1 + \mathcal{L}_{\mathrm{L},n+2}e_2 + \mathcal{L}_{\mathrm{L},n+3}e_3 \\
&= \psi^n + \phi^n + 1 + \left(\psi^{n+1}+\phi^{n+1}+1\right)e_1 + \left(\psi^{n+2}+\phi^{n+2}+1\right)e_2 \\
&\quad + \left(\psi^{n+3}+\phi^{n+3}+1\right)e_3 \\
&= \psi^n\left(1+\psi e_1+\psi^2 e_2+\psi^3 e_3\right) + \phi^n\left(1+\phi e_1+\phi^2 e_2+\phi^3 e_3\right) + Q^* \\
&= \psi^n\psi^* + \phi^n\phi^* + Q^*.
\end{aligned}$$

By (2.12) and (3.1), we obtain

$$\begin{aligned}
\mathcal{QM}_n &= \mathcal{M}_n + \mathcal{M}_{n+1}e_1 + \mathcal{M}_{n+2}e_2 + \mathcal{M}_{n+3}e_3 \\
&= \frac{\psi^{n+2}-\phi^{n+2}}{\psi-\phi} - 1 + \left(\frac{\psi^{n+3}-\phi^{n+3}}{\psi-\phi} - 1\right)e_1 \\
&\quad + \left(\frac{\psi^{n+4}-\phi^{n+4}}{\psi-\phi} - 1\right)e_2 + \left(\frac{\psi^{n+5}-\phi^{n+5}}{\psi-\phi} - 1\right)e_3 \\
&= \frac{\psi^{n+2}}{\psi-\phi}\left(1+\psi e_1\psi^2 e_2\psi^3 e_3\right) - \frac{\phi^{n+2}}{\psi-\phi}\left(1+\phi e_1\phi^2 e_2\phi^3 e_3\right) - Q^* \\
&= \frac{\psi^{n+2}\psi^* - \phi^{n+2}\phi^*}{\psi-\phi} - Q^*. \qquad \square
\end{aligned}$$

Remark 3.9. By the definitions of ψ^*, ϕ^* and Q^*, we get the following relationships:

$$\psi^*\phi^* = 1+\alpha-\beta+\gamma+\left(1-\sqrt{5}\frac{\Delta}{\alpha}\right)e_1+\left(3-\sqrt{5}\frac{\Delta}{\beta}\right)e_2+\left(4-\sqrt{5}\frac{\Delta}{\gamma}\right)e_3,$$

$$\phi^*\psi^* = 1+\alpha-\beta+\gamma+\left(1+\sqrt{5}\frac{\Delta}{\alpha}\right)e_1+\left(3+\sqrt{5}\frac{\Delta}{\beta}\right)e_2+\left(4-\sqrt{5}\frac{\Delta}{\gamma}\right)e_3,$$

$$\psi^*Q^* = -1-\psi\left(\alpha+\psi\beta+\psi^2\gamma\right)-\frac{\Delta}{\alpha}\psi e_1+\sqrt{5}\frac{\Delta}{\beta}\psi^2 e_2-\frac{\Delta}{\gamma}e_3+\psi^*+Q^*,$$
$$\phi^*Q^* = -1-\phi\left(\alpha+\phi\beta+\phi^2\gamma\right)-\frac{\Delta}{\alpha}\phi e_1+\sqrt{5}\frac{\Delta}{\beta}\phi^2 e_2-\frac{\Delta}{\gamma}e_3+\phi^*+Q^*,$$
$$Q^*\psi^* = Q^*+\phi^*-\left(1+\psi\alpha+\psi^2\beta+\psi^3\gamma\right)+\psi\frac{\Delta}{\alpha}e_1+(\psi+2)\frac{\Delta}{\beta}e_2+\frac{\Delta}{\gamma}e_3,$$
$$Q^*\phi^* = Q^*+\phi^*-\left(1+\phi\alpha+\phi^2\beta+\phi^3\gamma\right)+\phi\frac{\Delta}{\alpha}e_1-\frac{\Delta}{\beta}e_2+\frac{\Delta}{\gamma}e_3.$$

Theorem 3.10. *The exponential generating functions for LEQ, LLEQ and MLEQ are*

$$\sum_{n=0}^{\infty}\mathcal{QL}_n\frac{y^n}{n!} = 2\left(\frac{\psi\psi^*e^{\psi y}-\phi\phi^*e^{\phi y}}{\psi-\phi}\right)-Q^*e^y,$$
$$\sum_{n=0}^{\infty}\mathcal{QL}_{\mathrm{L},n}\frac{y^n}{n!} = \psi^*e^{\psi y}+\phi^*e^{\phi y}+Q^*e^y,$$
$$\sum_{n=0}^{\infty}\mathcal{QM}_n\frac{y^n}{n!} = \frac{\psi^{n+1}\psi^*e^{\psi y}-\phi^{n+1}\phi^*e^{\phi y}}{\psi-\phi}-Q^*e^y.$$

Proof. By formula (3.14), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{QL}_n\frac{y^n}{n!} &= \sum_{n=0}^{\infty}\left[2\left(\frac{\psi^{n+1}\psi^*-\phi^{n+1}\phi^*}{\psi-\phi}\right)-Q^*\right]\frac{y^n}{n!}\\ &= \frac{2\psi\psi^*}{\psi-\phi}\sum_{n=0}^{\infty}\frac{(\psi y)^n}{n!}-\frac{2\phi\phi^*}{\psi-\phi}\sum_{n=0}^{\infty}\frac{(\phi y)^n}{n!}-Q^*\sum_{n=0}^{\infty}\frac{y^n}{n!}\\ &= 2\left(\frac{\psi\psi^*e^{\psi y}-\phi\phi^*e^{\phi y}}{\psi-\phi}\right)-Q^*e^y.\end{aligned}$$

Other equations can be proved in the same way. □

Theorem 3.11. *The Vajda's identity for the LEQ is*

$$\begin{aligned}&\mathcal{QL}_{n+r}\mathcal{QL}_{n+s}-\mathcal{QL}_n\mathcal{QL}_{n+r+s}\\ &\quad= \frac{2\sqrt{5}}{5}\big(2(-1)^{n+1}\mathrm{F}_r\left(\psi^s\phi^*\psi^*-\phi^s\psi^*\phi^*\right)\\ &\qquad-\left(\psi^n\psi^*-\phi^n\phi^*\right)Q^*+Q^*\left(\psi^{n+s}\psi^*-\phi^{n+s}\phi^*\right)\big).\end{aligned}$$

Proof. Let $\mathcal{VL}$ be the Vajda's identity for LEQ. By the Binet formula for LEQ and the Binet formula for Fibonacci numbers, we get

$$\begin{aligned}
\mathcal{VL} &= \mathcal{QL}_{n+r}\mathcal{QL}_{n+s} - \mathcal{QL}_n\mathcal{QL}_{n+r+s} \\
&= \left(2\frac{\psi^{n+r+1}\psi^* - \phi^{n+r+1}\phi^*}{\psi-\phi} - Q^*\right)\left(2\frac{\psi^{n+s+1}\psi^* - \phi^{n+s+1}\phi^*}{\psi-\phi} - Q^*\right) \\
&\quad - \left(2\frac{\psi^{n+1}\psi^* - \phi^{n+1}\phi^*}{\psi-\phi} - Q^*\right)\left(2\frac{\psi^{n+r+s+1}\psi^* - \phi^{n+r+s+1}\phi^*}{\psi-\phi} - Q^*\right) \\
&= \frac{4}{(\psi-\phi)^2}\big(\psi^{2n+r+s+2}(\psi^*)^2 - \psi^{n+r+1}\phi^{n+s+1}\psi^*\phi^* \\
&\qquad\qquad -\psi^{n+s+1}\phi^{n+r+1}\phi^*\psi^* + \phi^{2n+r+s+2}(\phi^*)^2\big) \\
&\quad - \frac{2}{\psi-\phi}\left(\psi^{n+r+1}\psi^* - \phi^{n+r+1}\phi^* + \psi^{n+1}\psi^* - \phi^{n+1}\phi^*\right)Q^* \\
&\quad + \frac{2Q^*}{\psi-\phi}\left(\psi^{n+r+s+1}\psi^* - \phi^{n+r+s+1}\phi^* - \psi^{n+s+1}\psi^* + \phi^{n+s+1}\phi^*\right) \\
&= \frac{4}{(\psi-\phi)^2}\Big(\left(\psi^{n+1}\phi^{n+r+s+1} - \psi^{n+r+1}\phi^{n+s+1}\right)\psi^*\phi^* \\
&\qquad\qquad + \left(\psi^{n+r+s+1}\phi^{n+1} - \psi^{n+s+1}\phi^{n+r+1}\right)\phi^*\psi^*\Big) \\
&\quad + \frac{2}{\psi-\phi}\left(\psi^{n+1}(1-\psi)\psi^* - \phi^{n+1}(1-\phi)\phi^*\right)Q^* \\
&\quad + \frac{2Q^*}{\psi-\phi}\left(\psi^{n+s+1}(\psi-1)\psi^* - \phi^{n+s+1}(\phi-1)\phi^*\right) \\
&= \frac{4}{5}(-1)^{n+1}\left(\phi^n(\phi^r-\psi^r)\psi^*\phi^* + \psi^n(\psi^r-\phi^r)\phi^*\psi^*\right) \\
&\quad + \frac{2}{\sqrt{5}}\big(\psi\phi(\psi^n\psi^* - \phi^n\phi^*)Q^* + Q^*\psi\phi\left(\phi^{n+s}\phi^* - \alpha^{n+s}\psi^*\right)\big) \\
&= \frac{2\sqrt{5}}{5}\big(2(-1)^{n+1}\mathrm{F}_r(\psi^s\phi^*\psi^* - \phi^s\psi^*\phi^*) \\
&\qquad - (\psi^n\psi^* - \phi^n\phi^*)Q^* + Q^*\left(\psi^{n+s}\psi^* - \phi^{n+s}\phi^*\right)\big).
\end{aligned}$$

□

In the following corollaries we have particular cases of Vajda's identity.

Corollary 3.12. *For $r = -s$, we reduce it to Catalan's identity for LEQ as follows:*

$$\begin{aligned}
\mathcal{QL}_{n-s}\mathcal{QL}_{n+s} - \mathcal{QL}_n^2 = \frac{2\sqrt{5}}{5}\big(&2(-1)^{n+1}\mathrm{F}_r(\psi^s\phi^*\psi^* - \phi^s\psi^*\phi^*) \\
&- (\psi^n\psi^* - \phi^n\phi^*)Q^* + Q^*\left(\psi^{n+s}\psi^* - \phi^{n+s}\phi^*\right)\big).
\end{aligned}$$

Corollary 3.13. *For $s=-r=1$, we reduce it to Cassini identity for LEQ as follows:*

$$\begin{aligned}\mathcal{QL}_{n-1}\mathcal{QL}_{n+1}-\mathcal{QL}_n^2=\frac{2\sqrt{5}}{5}\big(2\,(-1)^{n+1}\,(\psi\phi^*\psi^*-\phi\psi^*\phi^*)\\ -(\psi^n\psi^*-\phi^n\phi^*)\,Q^*+Q^*\big(\psi^{n+1}\psi^*-\phi^{n+1}\phi^*\big)\big).\end{aligned}$$

Corollary 3.14. *For $s\longrightarrow m-n$, $m\geq n$ and $r=1$, we reduce it to d'Ocagne's identity for LEQ as follows:*

$$\begin{aligned}\mathcal{QL}_{n+1}\mathcal{QL}_m-\mathcal{QL}_n\mathcal{QL}_{m+1}=\frac{2\sqrt{5}}{5}\big(2\,\big(\psi^{m-n}\phi^*\psi^*-\phi^{m-n}\psi^*\phi^*\big)\\ -(\psi^n\psi^*-\phi^n\phi^*)\,Q^*+Q^*(\psi^m\psi^*-\phi^m\phi^*)\big).\end{aligned}$$

Theorem 3.15. *The Vajda's identity for the LLEQ is*

$$\begin{aligned}&\mathcal{QL}_{\mathrm{L},n+r}\mathcal{QL}_{\mathrm{L},n+s}-\mathcal{QL}_{\mathrm{L},n}\mathcal{QL}_{\mathrm{L},n+r+s}\\ &\quad=\sqrt{5}\,(-1)^n\,\phi^s\mathrm{F}_r\,(\phi^s\psi^*\phi^*-\psi^s\phi^*\psi^*)+\psi^n\,(\psi^r-1)\,(\psi^*Q^*-\psi^sQ^*\psi^*)\\ &\quad\quad+\phi^n\,(\phi^r-1)\,(\phi^*Q^*-\phi^sQ^*\phi^*)\,.\end{aligned}$$

Proof. Let $\mathcal{VL}_{\mathrm{L}}$ be the Vajda's identity for LLEQ. By the Binet formula for LLEQ and the Binet formula for Fibonacci numbers, we get

$$\begin{aligned}\mathcal{VL}_{\mathrm{L}}&=\mathcal{QL}_{\mathrm{L},n+r}\mathcal{QL}_{\mathrm{L},n+s}-\mathcal{QL}_{\mathrm{L},n}\mathcal{QL}_{\mathrm{L},n+r+s}\\ &=\big(\psi^{n+r}\psi^*+\phi^{n+r}\phi^*+Q^*\big)\big(\psi^{n+s}\psi^*+\phi^{n+s}\phi^*+Q^*\big)\\ &\quad\;-(\psi^n\psi^*+\phi^n\phi^*+Q^*)\big(\psi^{n+r+s}\psi^*+\phi^{n+r+s}\phi^*+Q^*\big)\\ &=\big(\psi^{n+r}\phi^{n+s}-\psi^n\phi^{n+r+s}\big)\,\psi^*\phi^*+\big(\psi^{n+s}\phi^{n+r}-\psi^{n+r+s}\phi^n\big)\,\phi^*\psi^*\\ &\quad+\big(\psi^{n+r}-\psi^n\big)\,\psi^*Q^*+\big(\psi^{n+s}-\psi^{n+r+s}\big)\,Q^*\psi^*\\ &\quad+\big(\phi^{n+r}-\phi^n\big)\,\phi^*Q^*+\big(\phi^{n+s}-\phi^{n+r+s}\big)\,Q^*\phi^*\\ &=\psi^n\phi^{n+s}\,(\psi^r-\phi^r)\,\psi^*\phi^*+\psi^{n+s}\phi^n\,(\phi^r-\psi^r)\,\phi^*\psi^*+\psi^n\,(\psi^r-1)\,\psi^*Q^*\\ &\quad+\psi^{n+s}\,(1-\psi^r)\,Q^*\psi^*+\phi^n\,(\phi^r-1)\,\phi^*Q^*+\phi^{n+s}\,(1-\phi^r)\,Q^*\phi^*\\ &=\sqrt{5}\,(-1)^n\,\phi^s\mathrm{F}_r\psi^*\phi^*-\sqrt{5}\,(-1)^n\,\psi^s\mathrm{F}_r\phi^*\psi^*+\psi^n\,(\psi^r-1)\,\psi^*Q^*\\ &\quad+\psi^{n+s}\,(1-\psi^r)\,Q^*\psi^*+\phi^n\,(\phi^r-1)\,\phi^*Q^*+\phi^{n+s}\,(1-\phi^r)\,Q^*\phi^*\\ &=\sqrt{5}\,(-1)^n\,\phi^s\mathrm{F}_r\,(\phi^s\psi^*\phi^*-\psi^s\phi^*\psi^*)+\psi^n\,(\psi^r-1)\,(\psi^*Q^*-\psi^sQ^*\psi^*)\\ &\quad+\phi^n\,(\phi^r-1)\,(\phi^*Q^*-\phi^sQ^*\phi^*)\,.\end{aligned}$$

□

In the following corollaries we have particular cases of Vajda's identity.

Corollary 3.16. *For $r = -s$, we reduce it to Catalan's identity for LLEQ as follows:*

$$\begin{aligned}&\mathcal{QL}_{\mathrm{L},n-s}\mathcal{QL}_{\mathrm{L},n+s} - \mathcal{QL}^2_{\mathrm{L},n}\\ &\quad= \sqrt{5}\,(-1)^n\,\phi^s\left(\phi^s\psi^*\phi^* - \psi^s\phi^*\psi^*\right) + \psi^n\left(\psi^{-s}-1\right)\left(\psi^*Q^* - \psi^sQ^*\psi^*\right)\\ &\qquad+ \phi^n\left(\phi^{-s}-1\right)\left(\phi^*Q^* - \phi^sQ^*\phi^*\right).\end{aligned}$$

Corollary 3.17. *For $s = -r = 1$, we reduce it to Cassini identity for LLEQ as follows:*

$$\begin{aligned}&\mathcal{QL}_{\mathrm{L},n-1}\mathcal{QL}_{\mathrm{L},n+1} - \mathcal{QL}^2_{\mathrm{L},n}\\ &\quad= \sqrt{5}\,(-1)^n\,\phi\left(\phi\psi^*\phi^* - \psi\phi^*\psi^*\right) - \left(\frac{3-\sqrt{5}}{2}\right)\psi^n\left(\psi^*Q^* - \psi Q^*\psi^*\right)\\ &\qquad- \phi\left(\frac{3+\sqrt{5}}{2}\right)\left(\phi^*Q^* - \phi Q^*\phi^*\right).\end{aligned}$$

Corollary 3.18. *For $s \longrightarrow m-n$, $m \geq n$ and $r = 1$, we reduce it to d'Ocagne's identity for LLEQ as follows:*

$$\begin{aligned}&\mathcal{QL}_{\mathrm{L},n+1}\mathcal{QL}_{\mathrm{L},m} - \mathcal{QL}_{\mathrm{L},n}\mathcal{QL}_{\mathrm{L},m+1}\\ &\quad= \sqrt{5}\,(-1)^n\,\phi^{m-n}\left(\phi^{m-n}\psi^*\phi^* - \psi^{m-n}\phi^*\psi^*\right) + \psi^n\phi\left(\psi^*Q^* - \psi^{m-n}Q^*\psi^*\right)\\ &\qquad- \phi^n\psi\left(\phi^*Q^* - \phi^{m-n}Q^*\phi^*\right).\end{aligned}$$

Theorem 3.19. *The Vajda's identitiy for the MLEQ is*

$$\begin{aligned}&\mathcal{QM}_{n+r}\mathcal{QM}_{n+s} - \mathcal{QM}_n\mathcal{QM}_{n+r+s}\\ &\quad= (-1)^n\,\mathrm{F}_r\left(\psi^s\phi^*\psi^* - \phi^s\psi^*\phi^*\right) + \frac{1}{\sqrt{5}}\psi^{n+2}\left(1-\psi^r\right)\left(\psi^*Q^* - \psi^sQ^*\psi^*\right)\\ &\qquad- \frac{1}{\sqrt{5}}\phi^{n+2}\left(1-\phi^r\right)\left(\phi^*Q^* - \phi^sQ^*\phi^*\right).\end{aligned}$$

Proof. Let $\mathcal{VM}$ be the Vajda's identity for MLEQ. By the Binet formula for MLEQ and the Binet formula for Fibonacci numbers, we get

$$\begin{aligned}
\mathcal{VM} &= \mathcal{QM}_{n+r}\mathcal{QM}_{n+s} - \mathcal{QM}_n\mathcal{QM}_{n+r+s} \\
&= \left(\frac{\psi^{n+r+2}\psi^* - \phi^{n+r+2}\phi^*}{\psi-\phi} - Q^*\right)\left(\frac{\psi^{n+s+2}\psi^* - \phi^{n+s+2}\phi^*}{\psi-\phi} - Q^*\right) \\
&\quad - \left(\frac{\psi^{n+2}\psi^* - \phi^{n+2}\phi^*}{\psi-\phi} - Q^*\right)\left(\frac{\psi^{n+r+s+2}\psi^* - \phi^{n+r+s+2}\phi^*}{\psi-\phi} - Q^*\right) \\
&= \frac{1}{(\psi-\phi)^2}\big(\left(\psi^{n+2}\phi^{n+r+s+2} - \psi^{n+r+2}\phi^{n+s+2}\right)\psi^*\phi^* \\
&\qquad\qquad + \left(\psi^{n+r+s+2}\phi^{n+2} - \psi^{n+s+2}\phi^{n+r+2}\right)\phi^*\psi^*\big) \\
&\quad + \frac{1}{\psi-\phi}\left(\psi^{n+2}\left(1-\psi^r\right)\psi^* + \phi^{n+2}\left(\phi^r-1\right)\phi^*\right)Q^* \\
&\quad + \frac{Q^*}{\psi-\phi}\left(\psi^{n+s+2}\left(\psi^r-1\right)\psi^* + \phi^{n+s+2}\left(1-\phi^r\right)\phi^*\right) \\
&= \frac{\psi^r\phi^r}{(\psi-\phi)^2}\psi^{n+2}\phi^{n+2}\left(\psi^s\phi^*\psi^* - \phi^s\psi^*\phi^*\right) \\
&\quad + \frac{1-\psi^r}{\psi-\phi}\left(\psi^{n+2}\psi^*Q^* - \psi^{n+s+2}Q^*\psi^*\right) \\
&\quad + \frac{1-\phi^r}{\psi-\phi}\left(\phi^{n+2}Q^*\phi^* - \psi^{n+2}\phi^*Q^*\right) \\
&= (-1)^n\,\mathrm{F}_r\left(\psi^s\phi^*\psi^* - \phi^s\psi^*\phi^*\right) + \frac{1}{\sqrt{5}}\psi^{n+2}\left(1-\psi^r\right)\left(\psi^*Q^* - \psi^sQ^*\psi^*\right) \\
&\quad - \frac{1}{\sqrt{5}}\phi^{n+2}\left(1-\phi^r\right)\left(\phi^*Q^* - \phi^sQ^*\phi^*\right).
\end{aligned}$$ □

In the following corollaries we have particular cases of Vajda's identity.

Corollary 3.20. *For $r=-s$, we reduce it to Catalan's identity for MLEQ as follows:*

$$\begin{aligned}
\mathcal{QM}_{n-s}\mathcal{QM}_{n+s} - \mathcal{QM}_n^2 &= (-1)^{n+s+3}\,\mathrm{F}_s\left(\psi^s\phi^*\psi^* - \phi^s\psi^*\phi^*\right) \\
&\quad + \frac{1}{\sqrt{5}}\psi^{n+2}\left(1-\psi^{-s}\right)\left(\psi^*Q^* - \psi^sQ^*\psi^*\right) \\
&\quad - \frac{1}{\sqrt{5}}\phi^{n+2}\left(1-\phi^{-s}\right)\left(\phi^*Q^* - \phi^sQ^*\phi^*\right).
\end{aligned}$$

Corollary 3.21. *For $s = -r = 1$, we reduce it to Cassini identity for MLEQ as follows:*

$$\begin{aligned}\mathcal{QM}_{n-1}\mathcal{QM}_{n+1} - \mathcal{QM}_n^2 = (-1)^n \left(\psi\phi^*\psi^* - \phi\psi^*\phi^*\right) \\ + \frac{1}{\sqrt{5}}\psi^{n+2}\left(1-\psi^{-1}\right)\left(\psi^*Q^* - \psi Q^*\psi^*\right) \\ - \frac{1}{\sqrt{5}}\phi^{n+2}\left(1-\phi^{-1}\right)\left(\phi^*Q^* - \phi Q^*\phi^*\right).\end{aligned}$$

Corollary 3.22. *For $s \longrightarrow m-n$, $m \geq n$ and $r = 1$, we it reduce to d'Ocagne's identity for MLEQ as follows:*

$$\begin{aligned}\mathcal{QM}_{n+1}\mathcal{QM}_m - \mathcal{QM}_n\mathcal{QM}_{m+1} = (-1)^n \left(\psi^{m-n}\phi^*\psi^* - \phi^{m-n}\psi^*\phi^*\right) \\ + \frac{1}{\sqrt{5}}\psi^{n+2}\phi\left(\psi^*Q^* - \psi^{m-n}Q^*\psi^*\right) \\ - \frac{1}{\sqrt{5}}\phi^{n+2}\psi\left(\phi^*Q^* - \phi^{m-n}Q^*\phi^*\right).\end{aligned}$$

Corollary 3.23. *Considering* (2.13) *and* (2.14)*, we obtain*

$$\begin{aligned}2\mathcal{QM}_n &= 2\left(\mathcal{M}_n + \mathcal{M}_{n+1}e_1 + \mathcal{M}_{n+2}e_2 + \mathcal{M}_{n+3}e_3\right) \\ &= 2\mathcal{L}_{n+1} - 1 + \left(2\mathcal{L}_{n+2} - 1\right)e_1 + \left(2\mathcal{L}_{n+3} - 1\right)e_2 + \left(2\mathcal{L}_{n+4} - 1\right)e_3 \\ &= 2\mathcal{QL}_{n+1} - Q^*,\end{aligned}$$

$$\begin{aligned}2\mathcal{QL}_{\mathrm{L},n} &= 2\left(\mathcal{L}_{\mathrm{L},n} + \mathcal{L}_{\mathrm{L},n+1}e_1 + \mathcal{L}_{\mathrm{L},n+2}e_2 + \mathcal{L}_{\mathrm{L},n+3}e_3\right) \\ &= \mathcal{L}_{\mathrm{L},n-2} + \mathcal{L}_{\mathrm{L},n} + 2 + \left(\mathcal{L}_{\mathrm{L},n-1} + \mathcal{L}_{\mathrm{L},n+1} + 2\right)e_1 \\ &\quad + \left(\mathcal{L}_{\mathrm{L},n} + \mathcal{L}_{\mathrm{L},n+2} + 2\right)e_2 + \left(\mathcal{L}_{\mathrm{L},n+1} + \mathcal{L}_{\mathrm{L},n+3} + 2\right)e_3 \\ &= 2\mathcal{QL}_{n-2} + \mathcal{QL}_n + 2Q^*\end{aligned}$$

and

$$\begin{aligned}\mathcal{QM}_n &= \mathcal{M}_n + \mathcal{M}_{n+1}e_1 + \mathcal{M}_{n+2}e_2 + \mathcal{M}_{n+3}e_3 \\ &= \mathcal{L}_{\mathrm{L},n} + \mathcal{L}_{\mathrm{L},n-1} - \mathrm{F}_n - 3 + \left(\mathcal{L}_{\mathrm{L},n+1} + \mathcal{L}_{\mathrm{L},n} - \mathrm{F}_{n+1} - 3\right)e_1 \\ &\quad + \left(\mathcal{L}_{\mathrm{L},n+2} + \mathcal{L}_{\mathrm{L},n+1} - \mathrm{F}_{n+2} - 3\right)e_2 \\ &\quad + \left(\mathcal{L}_{\mathrm{L},n+3} + \mathcal{L}_{\mathrm{L},n+2} - \mathrm{F}_{n+3} - 3\right)e_3 \\ &= \mathcal{QL}_{\mathrm{L},n} + \mathcal{QL}_{\mathrm{L},n-1} - \mathcal{Q}\mathrm{F}_n - 3Q^*,\end{aligned}$$

where $\mathcal{Q}\mathrm{F}_n$ is the nth Fibonacci elliptic quaternion.

4. The matrix representations of LEQ, LLEQ and MLEQ. In addition to previous formulations, we can define the matrix equalities for LEQ, LLEQ, and MLEQ using the following expressions.

Theorem 4.1. *Let $\mathcal{QL}_n$, $\mathcal{QL}_{\mathrm{L},n}$ and $\mathcal{QM}_n$ be the nth LEQ, LLEQ and MLEQ, respectively. For every $n > 0$, the following matrix equalities hold:*

$$\begin{bmatrix} \mathcal{QL}_3 & \mathcal{QL}_2 & \mathcal{QL}_1 \\ \mathcal{QL}_2 & \mathcal{QL}_1 & \mathcal{QL}_0 \\ \mathcal{QL}_1 & \mathcal{QL}_0 & \mathcal{QL}_{-1} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^n = \begin{bmatrix} \mathcal{QL}_{n+3} & \mathcal{QL}_{n+2} & \mathcal{QL}_{n+1} \\ \mathcal{QL}_{n+2} & \mathcal{QL}_{n+1} & \mathcal{QL}_n \\ \mathcal{QL}_{n+1} & \mathcal{QL}_n & \mathcal{QL}_{n-1} \end{bmatrix},$$

$$\begin{bmatrix} \mathcal{QL}_{\mathrm{L},3} & \mathcal{QL}_{\mathrm{L},2} & \mathcal{QL}_{\mathrm{L},1} \\ \mathcal{QL}_{\mathrm{L},2} & \mathcal{QL}_{\mathrm{L},1} & \mathcal{QL}_{\mathrm{L},0} \\ \mathcal{QL}_{\mathrm{L},1} & \mathcal{QL}_{\mathrm{L},0} & \mathcal{QL}_{\mathrm{L},-1} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^n = \begin{bmatrix} \mathcal{QL}_{\mathrm{L},n+3} & \mathcal{QL}_{\mathrm{L},n+2} & \mathcal{QL}_{\mathrm{L},n+1} \\ \mathcal{QL}_{\mathrm{L},n+2} & \mathcal{QL}_{\mathrm{L},n+1} & \mathcal{QL}_{\mathrm{L},n} \\ \mathcal{QL}_{\mathrm{L},n+1} & \mathcal{QL}_{\mathrm{L},n} & \mathcal{QL}_{\mathrm{L},n-1} \end{bmatrix},$$

$$\begin{bmatrix} \mathcal{QM}_3 & \mathcal{QM}_2 & \mathcal{QM}_1 \\ \mathcal{QM}_2 & \mathcal{QM}_1 & \mathcal{QM}_0 \\ \mathcal{QM}_1 & \mathcal{QM}_0 & \mathcal{QM}_{-1} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^n = \begin{bmatrix} \mathcal{QM}_{n+3} & \mathcal{QM}_{n+2} & \mathcal{QM}_{n+1} \\ \mathcal{QM}_{n+2} & \mathcal{QM}_{n+1} & \mathcal{QM}_n \\ \mathcal{QM}_{n+1} & \mathcal{QM}_n & \mathcal{QM}_{n-1} \end{bmatrix}$$

and

$$\begin{bmatrix} \mathcal{QL}_{n+2} \\ \mathcal{QL}_{n+1} \\ \mathcal{QL}_n \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} \mathcal{QL}_2 \\ \mathcal{QL}_1 \\ \mathcal{QL}_0 \end{bmatrix},$$

$$\begin{bmatrix} \mathcal{QL}_{\mathrm{L},n+2} \\ \mathcal{QL}_{\mathrm{L},n+1} \\ \mathcal{QL}_{\mathrm{L},n} \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} \mathcal{QL}_{\mathrm{L},2} \\ \mathcal{QL}_{\mathrm{L},1} \\ \mathcal{QL}_{\mathrm{L},0} \end{bmatrix},$$

$$\begin{bmatrix} \mathcal{QM}_{n+2} \\ \mathcal{QM}_{n+1} \\ \mathcal{QM}_n \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} \mathcal{QM}_2 \\ \mathcal{QM}_1 \\ \mathcal{QM}_0 \end{bmatrix}.$$

Proof. The proof is obvious by mathematical induction on n. The equality hold for $n = 1$. Now suppose that the equality is true for $n > 1$. Then we can verify it for $n + 1$ as follows

$$\begin{aligned} &\begin{bmatrix} \mathcal{QL}_3 & \mathcal{QL}_2 & \mathcal{QL}_1 \\ \mathcal{QL}_2 & \mathcal{QL}_1 & \mathcal{QL}_0 \\ \mathcal{QL}_1 & \mathcal{QL}_0 & \mathcal{QL}_{-1} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^{n+1} \\ &= \begin{bmatrix} \mathcal{QL}_3 & \mathcal{QL}_2 & \mathcal{QL}_1 \\ \mathcal{QL}_2 & \mathcal{QL}_1 & \mathcal{QL}_0 \\ \mathcal{QL}_1 & \mathcal{QL}_0 & \mathcal{QL}_{-1} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^n \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{QL}_{n+3} & \mathcal{QL}_{n+2} & \mathcal{QL}_{n+1} \\ \mathcal{QL}_{n+2} & \mathcal{QL}_{n+1} & \mathcal{QL}_n \\ \mathcal{QL}_{n+1} & \mathcal{QL}_n & \mathcal{QL}_{n-1} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{QL}_{n+4} & \mathcal{QL}_{n+3} & \mathcal{QL}_{n+2} \\ \mathcal{QL}_{n+3} & \mathcal{QL}_{n+2} & \mathcal{QL}_{n+1} \\ \mathcal{QL}_{n+2} & \mathcal{QL}_{n+1} & \mathcal{QL}_n \end{bmatrix}. \end{aligned}$$

□

Corollary 4.2. *Consider $\mathcal{QL}_n, \mathcal{QL}_m \in \mathbb{LH}$. We have the following:*

$$\begin{aligned} \varphi_{\mathcal{QL}_n} &: \mathbb{LH} \longrightarrow \mathbb{LH} \\ &\mathcal{QL}_m \longmapsto \varphi_{\mathcal{QL}_n}(\mathcal{QL}_m) = \mathcal{QL}_n \mathcal{QL}_m \end{aligned}$$

and

$$\begin{aligned} \mu_{\mathcal{QL}_n} &: \mathbb{LH} \longrightarrow \mathbb{LH} \\ &\mathcal{QL}_m \longmapsto \mu_{\mathcal{QL}_n}(\mathcal{QL}_m) = \mathcal{QL}_m \mathcal{QL}_n, \end{aligned}$$

where $\varphi_{\mathcal{QL}_n}$ and $\mu_{\mathcal{QL}_n}$ are respectively given as

$$\mathscr{L}_{\mathcal{QL}_n} = \begin{bmatrix} \mathcal{L}_n & -\alpha\mathcal{L}_{n+1} & -\beta\mathcal{L}_{n+2} & -\gamma\mathcal{L}_{n+3} \\ \mathcal{L}_{n+1} & \mathcal{L}_n & -\frac{\Delta}{\alpha}\mathcal{L}_{n+3} & \frac{\Delta}{\alpha}\mathcal{L}_{n+2} \\ \mathcal{L}_{n+2} & \frac{\Delta}{\beta}\mathcal{L}_{n+3} & \mathcal{L}_n & -\frac{\Delta}{\beta}\mathcal{L}_{n+1} \\ \mathcal{L}_{n+3} & -\frac{\Delta}{\gamma}\mathcal{L}_{n+2} & \frac{\Delta}{\gamma}\mathcal{L}_{n+1} & \mathcal{L}_n \end{bmatrix},$$

$$\mathscr{R}_{\mathcal{QL}_n} = \begin{bmatrix} \mathcal{L}_n & -\alpha\mathcal{L}_{n+1} & -\beta\mathcal{L}_{n+2} & -\gamma\mathcal{L}_{n+3} \\ \mathcal{L}_{n+1} & \mathcal{L}_n & \frac{\Delta}{\alpha}\mathcal{L}_{n+3} & -\frac{\Delta}{\alpha}\mathcal{L}_{n+2} \\ \mathcal{L}_{n+2} & -\frac{\Delta}{\beta}\mathcal{L}_{n+3} & \mathcal{L}_n & \frac{\Delta}{\beta}\mathcal{L}_{n+1} \\ \mathcal{L}_{n+3} & \frac{\Delta}{\gamma}\mathcal{L}_{n+2} & -\frac{\Delta}{\gamma}\mathcal{L}_{n+1} & \mathcal{L}_n \end{bmatrix}.$$

Other representations for the numbers in $\mathbb{LH}_{\mathrm{L}}$ and $\mathbb{MH}$ can be shown in the same way.

5. Conclusions. Elliptic quaternions defined in [10] has been given for a special case by replacing real number coefficients by Leonardo, Leonardo–Lucas and modified Leonardo number coefficients. We first defined elementary operations and its 4×4 matrix representations on the set of elliptic quaternions. Then we gave the definitions and relationships of Leonardo, Leonardo–Lucas and modified Leonardo numbers. We then went on to investigate relationships, recurrence relation and Binet formula of Leonardo, Leonardo–Lucas and modified Leonardo elliptic quaternions. We also provided the generating function. We finally investigated Vajda's identity and gave some corollaries for special cases such as Catalan, Cassini and D'Ocagne's identities. Lastly, we studied two different matrix representations of these numbers.

In the following table (Table 2), we show the numbers which are mentioned in this paper.

Number	Definition
Elliptic Quaternions ($\mathbb{H}_{\alpha,\beta,\gamma}$) [10]	$Q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3$: $q_0, q_1, q_2, q_3 \in \mathbb{R}$
nth Fibonacci Numbers	$\mathrm{F}_n = \mathrm{F}_{n-1} + \mathrm{F}_{n-2},\ \mathrm{F}_0 = 0,\ \mathrm{F}_1 = 1$
nth Lucas Numbers	$\mathrm{L}_n = \mathrm{L}_{n-1} + \mathrm{L}_{n-2},\ \mathrm{L}_0 = 2,\ \mathrm{L}_1 = 1$
nth Leonardo Numbers	$\mathcal{L}_n = \mathcal{L}_{n-1} + \mathcal{L}_{n-2} + 1$, $n \geq 2,\ \mathcal{L}_0 = \mathcal{L}_1 = 1$
nth Leonardo-Lucas Numbers	$\mathcal{L}_{\mathrm{L},n} = \mathcal{L}_{\mathrm{L},n-1} + \mathcal{L}_{\mathrm{L},n-2} + 1$, $n \geq 2,\ \mathcal{L}_{\mathrm{L},0} = 3,\ \mathcal{L}_{\mathrm{L},1} = 2$
nth modified Leonardo Numbers	$\mathcal{M}_n = \mathcal{M}_{n-1} + \mathcal{M}_{n-2} + 1$, $n \geq 2,\ \mathcal{M}_0 = 0,\ \mathcal{M}_1 = 1$
nth Fibonacci Elliptic Quaternions	$\mathcal{Q}\mathrm{F}_n = \mathrm{F}_n + \mathrm{F}_{n+1}e_1 + \mathrm{F}_{n+2}e_2 + \mathrm{F}_{n+3}e_3$
nth Lucas Elliptic Quaternions	$\mathcal{Q}\mathrm{L}_n = \mathrm{L}_n + \mathrm{L}_{n+1}e_1 + \mathrm{L}_{n+2}e_2 + \mathrm{L}_{n+3}e_3$
nth Leonardo Elliptic Quaternions ($\mathbb{LH}$)	$\mathcal{QL}_n = 2\mathcal{QL}_{n-1} - \mathcal{QL}_{n-3}$
nth Leonardo-Lucas Elliptic Quaternions ($\mathbb{LH}_\mathrm{L}$)	$\mathcal{QL}_{\mathrm{L},n} = 2\mathcal{QL}_{\mathrm{L},n-1} - \mathcal{QL}_{\mathrm{L},n-3}$
nth Modified-Leonardo Elliptic Quaternions ($\mathbb{MH}$)	$\mathcal{QM}_n = 2\mathcal{QM}_{n-1} - \mathcal{QM}_{n-3}$

Table 2. The numbers which are mentioned in this paper.

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