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UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

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ANETTA SZYNAL-LIANA

On non-Newtonian balancing type numbers

ABSTRACT. In this paper, we introduce non-Newtonian balancing type numbers. In non-Newtonian calculus, we examine formulas and identities for classical balancing numbers. We give Binet-type formula for non-Newtonian balancing numbers and the general bilinear index-reduction formula which implies Catalan, Cassini and d'Ocagne identities. Moreover, we give the generating function for balancing numbers in terms of non-Newtonian calculus.

1. Introduction. In [8], Grossman and Katz established a new family of calculi, called non-Newtonian calculi. Non-Newtonian calculus is an alternative to the usual calculus of Newton and Leibniz. Let α be a bijection between the set of real numbers $\mathbb{R}$ and a subset $\mathbb{R}_\alpha$ of $\mathbb{R}$. By α-arithmetic, we mean arithmetic whose operations are defined as follows:

α-addition	$x \dot{+} y = \alpha\left(\alpha^{-1}(x) + \alpha^{-1}(y)\right)$,
α-subtraction	$x \dot{-} y = \alpha\left(\alpha^{-1}(x) - \alpha^{-1}(y)\right)$,
α-multiplication	$x \dot{\times} y = \alpha\left(\alpha^{-1}(x) \cdot \alpha^{-1}(y)\right)$,
α-division	$x \dot{/} y = \frac{x}{y}\alpha = \alpha\left(\alpha^{-1}(x) : \alpha^{-1}(y)\right),\ \alpha^{-1}(y) \neq 0$,
α-order	$x \dot{<} y \ \Leftrightarrow\ \alpha^{-1}(x) < \alpha^{-1}(y)$.

If $x \in \mathbb{R}_\alpha$ and $\dot{0} \dot{<} \dot{x}$ (or $\dot{x} \dot{<} \dot{0}$), then we say that x is a α-positive number (or α-negative number). The number $x \dot{\times} x$ is called the α-square of x, denoted by $x^{\dot{2}}$, see [5]. If the exponent is the sum of several numbers, we

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will write $x^{(n)^{\cdot}}$ instead of $x^{\dot{n}}$, for example x to the power $n+1$ will be written as $x^{(n+1)^{\cdot}}$. Moreover, $\alpha(-x) = \alpha\left(-\alpha^{-1}(\dot{x})\right) = \dot{-}\dot{x}$ and $\dot{\sqrt{x}} = \alpha\left(\sqrt{\alpha^{-1}(x)}\right)$.

The function α is called a generator. Each generator generates exactly one arithmetic, and each arithmetic is generated by exactly one generator. The identity function generates classical arithmetic and the exponential function generates geometric arithmetic. Every property in the classical calculus has an analog in non-Newtonian calculus. Non-Newtonian calculi are useful mathematical tools in science, mathematics, engineering, economics etc., see e.g. [2, 3, 5, 6, 7, 9, 14].

A positive integer n is a balancing number if it is the solution of the Diophantine equation $1+2+\cdots+(n-1) = (n+1)+(n+2)+\cdots+(n+r)$ for some positive integer r, called the balancer corresponding to the balancing number n. The sequence $\{B_n\}$ of the balancing numbers was introduced by Behera and Panda in [1]. As mentioned in [1], n is a balancing number if and only if n^2 is a triangular number, i.e. $8n^2 + 1$ is a perfect square.

In [12], Panda introduced the sequence of Lucas-balancing numbers, denoted by $\{C_n\}$ and defined as follows: if B_n is a balancing number, the number C_n for which $(C_n)^2 = 8B_n^2 + 1$ is called a Lucas-balancing number.

The cobalancing numbers were defined and introduced in [13]. The authors called the positive integer number n a cobalancing number with the cobalancer r if $1 + 2 + \cdots + n = (n + 1) + (n + 2) + \cdots + (n + r)$. Let b_n denote the nth cobalancing number. The nth Lucas-cobalancing number c_n is defined with $(c_n)^2 = 8b_n^2 + 8b_n + 1$, see [10, 11].

In this paper, we introduce and study non-Newtonian balancing type numbers.

2. Main results. The balancing, Lucas-balancing, cobalancing and Lucas-cobalancing numbers can also be defined using the following recurrence relations

$$\begin{aligned} B_n &= 6B_{n-1} - B_{n-2} \text{ for } n \geq 2, \text{ with } B_0 = 0, B_1 = 1, \\ C_n &= 6C_{n-1} - C_{n-2} \text{ for } n \geq 2, \text{ with } C_0 = 1, C_1 = 3, \\ b_n &= 6b_{n-1} - b_{n-2} + 2 \text{ for } n \geq 2, \text{ with } b_0 = 0, b_1 = 0, \\ c_n &= 6c_{n-1} - c_{n-2} \text{ for } n \geq 2, \text{ with } c_0 = -1, c_1 = 1. \end{aligned}$$

The above-mentioned sequences can also be written using explicit formulas, called Binet type formulas, of the form

$$B_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}, \qquad C_n = \frac{\gamma^n + \delta^n}{2},$$

$$b_n = \frac{\gamma^{n-\frac{1}{2}} - \delta^{n-\frac{1}{2}}}{\gamma - \delta} - \frac{1}{2}, \qquad c_n = \frac{\gamma^{n-\frac{1}{2}} + \delta^{n-\frac{1}{2}}}{2},$$

for $n \geq 0$, where

$$\gamma = 3 + \sqrt{8},\ \delta = 3 - \sqrt{8},\ \gamma^{\frac{1}{2}} = 1 + \sqrt{2},\ \delta^{\frac{1}{2}} = 1 - \sqrt{2}.$$

For $n \geq 0$, the nth non-Newtonian balancing number, the nth non-Newtonian Lucas-balancing number, the nth non-Newtonian cobalancing number and the nth non-Newtonian Lucas-cobalancing number is defined by

$$NB_n = \dot{B_n} = \alpha(B_n),$$
$$NC_n = \dot{C_n} = \alpha(C_n),$$
$$Nb_n = \dot{b_n} = \alpha(b_n),$$
$$Nc_n = \dot{c_n} = \alpha(c_n),$$

respectively.

Tables 1–2 include initial terms of the balancing type numbers and non-Newtonian balancing type numbers, respectively.

n	0	1	2	3	4	5	6	7
B_n	0	1	6	35	204	1189	6930	40391
C_n	1	3	17	99	577	3363	19601	114243
b_n	0	0	2	14	84	492	2870	16730
c_n	-1	1	7	41	239	1393	8119	47321

Table 1. The balancing type numbers.

n	0	1	2	3	4	5	6	7
NB_n	$\alpha(0)$	$\alpha(1)$	$\alpha(6)$	$\alpha(35)$	$\alpha(204)$	$\alpha(1189)$	$\alpha(6930)$	$\alpha(40391)$
NC_n	$\alpha(1)$	$\alpha(3)$	$\alpha(17)$	$\alpha(99)$	$\alpha(577)$	$\alpha(3363)$	$\alpha(19601)$	$\alpha(114243)$
Nb_n	$\alpha(0)$	$\alpha(0)$	$\alpha(2)$	$\alpha(14)$	$\alpha(84)$	$\alpha(492)$	$\alpha(2870)$	$\alpha(16730)$
Nc_n	$\alpha(-1)$	$\alpha(1)$	$\alpha(7)$	$\alpha(41)$	$\alpha(239)$	$\alpha(1393)$	$\alpha(8119)$	$\alpha(47321)$

Table 2. The non-Newtonian balancing type numbers.

In the next part of this paper, we focus on the properties of non-Newtonian balancing numbers.

By the definition of non-Newtonian balancing numbers, we obtain the following recurrence relation

$$NB_n = \dot{6} \dot{\times} NB_{n-1} \dot{-} NB_{n-2} \text{ for } n \geq 2$$

with $NB_0 = \alpha(0)$, $NB_1 = \alpha(1)$.

For $n \geq 2$, we have

$$\begin{aligned}
\dot{6} \dot{\times} NB_{n-1} \dot{-} NB_{n-2} &= \alpha(6) \dot{\times} \alpha(B_{n-1}) \dot{-} \alpha(B_{n-2}) \\
&= \alpha\left(\alpha^{-1}(\alpha(6)) \cdot \alpha^{-1}(\alpha(B_{n-1}))\right) \dot{-} \alpha(B_{n-2}) \\
&= \alpha\left(6 \cdot B_{n-1}\right) \dot{-} \alpha(B_{n-2}) \\
&= \alpha\left(\alpha^{-1}(\alpha\left(6 \cdot B_{n-1}\right)) - \alpha^{-1}(\alpha(B_{n-2}))\right) \\
&= \alpha\left(6 \cdot B_{n-1} - B_{n-2}\right) = \alpha\left(B_n\right) = NB_n.
\end{aligned}$$

The next theorem gives an explicit formula for the nth non-Newtonian balancing number.

Theorem 1. *Let NB_n be the nth non-Newtonian balancing number. Then, for $n \geq 0$ the Binet-type formula is given by*

$$NB_n = \frac{\dot{\gamma}^{\dot{n}} \dot{-} \dot{\delta}^{\dot{n}}}{\dot{\gamma} \dot{-} \dot{\delta}}{}_{\alpha},$$

where $\dot{\gamma} = \dot{3} \dot{+} \dot{\sqrt{8}}$ and $\dot{\delta} = \dot{3} \dot{-} \dot{\sqrt{8}}$.

Proof. From the properties of non-Newtonian real numbers, we have

$$\frac{\dot{\gamma}^{\dot{n}} \dot{-} \dot{\delta}^{\dot{n}}}{\dot{\gamma} \dot{-} \dot{\delta}}{}_{\alpha} = \alpha\left(\frac{\alpha^{-1}\left(\dot{\gamma}^{\dot{n}} \dot{-} \dot{\delta}^{\dot{n}}\right)}{\alpha^{-1}\left(\dot{\gamma} \dot{-} \dot{\delta}\right)}\right)$$

$$= \alpha\left(\frac{\alpha^{-1}\left(\overbrace{\dot{\gamma} \dot{\times} \ldots \dot{\times} \dot{\gamma}}^{n\ times} \dot{-} \overbrace{\dot{\delta} \dot{\times} \ldots \dot{\times} \dot{\delta}}^{n\ times}\right)}{\alpha^{-1}\left(\alpha\left(\alpha^{-1}(\dot{\gamma}) - \alpha^{-1}(\dot{\delta})\right)\right)}\right)$$

$$= \alpha\left(\frac{\alpha^{-1}\left(\alpha\left(\overbrace{\alpha^{-1}(\dot{\gamma}) \cdot \ldots \cdot \alpha^{-1}(\dot{\gamma})}^{n\ times}\right) \dot{-} \alpha\left(\overbrace{\alpha^{-1}(\dot{\delta}) \cdot \ldots \cdot \alpha^{-1}(\dot{\delta})}^{n\ times}\right)\right)}{\alpha^{-1}\left(\alpha\left(\gamma - \delta\right)\right)}\right)$$

$$= \alpha\left(\frac{\alpha^{-1}\left(\alpha\left(\overbrace{\gamma \cdot \ldots \cdot \gamma}^{n\ times}\right) \dot{-} \alpha\left(\overbrace{\delta \cdot \ldots \cdot \delta}^{n\ times}\right)\right)}{\gamma - \delta}\right)$$

$$= \alpha\left(\frac{\alpha^{-1}\left(\alpha\left(\gamma^n\right) \dot{-} \alpha\left(\delta^n\right)\right)}{\gamma - \delta}\right) = \alpha\left(\frac{\alpha^{-1}\left(\alpha\left[\alpha^{-1}(\alpha\left(\gamma^n\right)) - \alpha^{-1}(\alpha\left(\delta^n\right))\right]\right)}{\gamma - \delta}\right)$$

$$= \alpha\left(\frac{\alpha^{-1}\left(\alpha\left[\gamma^n - \delta^n\right]\right)}{\gamma - \delta}\right) = \alpha\left(\frac{\gamma^n - \delta^n}{\gamma - \delta}\right) = \alpha(B_n) = NB_n,$$

which ends the proof. □

In the following theorem, we derive a general bilinear index-reduction formula for non-Newtonian balancing numbers.

Theorem 2. *Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ be integers such that $a+b = c+d$ and let NB_n be the nth non-Newtonian balancing number. Then*

$$NB_a \dot{\times} NB_b \dot{-} NB_c \dot{\times} NB_d = \alpha\left(B_a \cdot B_b - B_c \cdot B_d\right).$$

Proof. From the properties of non-Newtonian real numbers, we have

$$\begin{aligned} &NB_a \dot{\times} NB_b \dot{-} NB_c \dot{\times} NB_d \\ &\quad = \alpha(B_a) \dot{\times} \alpha(B_b) \dot{-} \alpha(B_c) \dot{\times} \alpha(B_d) \\ &\quad = \alpha\left(\alpha^{-1}(\alpha(B_a)) \cdot \alpha^{-1}(\alpha(B_b))\right) \dot{-} \alpha\left(\alpha^{-1}(\alpha(B_c)) \cdot \alpha^{-1}(\alpha(B_d))\right) \\ &\quad = \alpha\left(B_a \cdot B_b\right) \dot{-} \alpha\left(B_c \cdot B_d\right) \\ &\quad = \alpha\left(\alpha^{-1}(\alpha(B_a \cdot B_b)) - \alpha^{-1}(\alpha(B_c \cdot B_d))\right) \\ &\quad = \alpha\left(B_a \cdot B_b - B_c \cdot B_d\right), \end{aligned}$$

which ends the proof. □

For special values of a, b, c, d, using Theorem 2, we can obtain some identities for non-Newtonian balancing numbers, among others Catalan type identity (for $a = b = n$, $c = n + r$, $d = n - r$), Cassini type identity (for $a = b = n$, $c = n + 1$, $d = n - 1$), d'Ocagne type identity (for $a = m$, $b = n + 1$, $c = n$, $d = m + 1$). Let us recall the analogous identities for the classical balancing numbers, given in [4]

$$\begin{aligned} \text{Catalan identity} \quad & B_n^2 - B_{n+r}B_{n-r} = B_r^2, \\ \text{Cassini indentity} \quad & B_n^2 - B_{n+1}B_{n-1} = 1, \\ \text{d'Ocagne indentity} \quad & B_mB_{n+1} - B_nB_{m+1} = B_{m-n}. \end{aligned}$$

Corollary 3 (Catalan type identity for non-Newtonian balancing numbers). *Let $n \geq 0$, $r \geq 0$ be integers such that $n \geq r$. Then*

$$(NB_n)^{\dot{2}} \dot{-} NB_{n+r} \dot{\times} NB_{n-r} = \dot{B}_r \dot{\times} \dot{B}_r.$$

Proof. By Theorem 2 and the properties of non-Newtonian real numbers, we have

$$\begin{aligned} (NB_n)^{\dot{2}} \dot{-} NB_{n+r} \dot{\times} NB_{n-r} &= \alpha\left(B_n^2 - B_{n+r} \cdot B_{n-r}\right) = \alpha\left(B_r^2\right) \\ &= \alpha\left(B_r \cdot B_r\right) = \alpha\left(\alpha^{-1}(\dot{B}_r) \cdot \alpha^{-1}(\dot{B}_r)\right) = \dot{B}_r \dot{\times} \dot{B}_r, \end{aligned}$$

which ends the proof. □

The proofs of the next two corollaries are the same, so we omit them.

Corollary 4 (Cassini type identity for non-Newtonian balancing numbers). *Let $n \geq 1$ be an integer. Then*

$$(NB_n)^{\dot{2}} \dot{-} NB_{n+1} \dot{\times} NB_{n-1} = \dot{1}.$$

Corollary 5 (d'Ocagne type identity for non-Newtonian balancing numbers). *Let $n \geq 0$, $m \geq 0$ be integers such that $m \geq n$. Then*

$$NB_m \dot{\times} NB_{n+1} \dot{-} NB_n \dot{\times} NB_{m+1} = \dot{B}_{m-n}.$$

In the next theorem, we use the following formula (see [4])

$$\sum_{i=0}^{n} B_i = \frac{B_{n+1} - B_n - 1}{4}.$$

Theorem 6. *Let $n \geq 0$ be an integer. Then*

$${}_{\alpha}\sum_{i=0}^{n} NB_i = \frac{\dot{B}_{n+1} \dot{-} \dot{B}_n \dot{-} \dot{1}}{\dot{4}}_{\alpha}.$$

Proof. From the properties of non-Newtonian real numbers, we have

$$\begin{aligned}
{}_{\alpha}\sum_{i=0}^{n} NB_i &= NB_0 \dot{+} NB_1 \dot{+} \ldots \dot{+} NB_n \\
&= \alpha\left(\alpha^{-1}(NB_0) + \alpha^{-1}(NB_1) + \ldots + \alpha^{-1}(NB_n)\right) \\
&= \alpha\left(B_0 + B_1 + \ldots + B_n\right) = \alpha\left(\frac{B_{n+1} - B_n - 1}{4}\right) \\
&= \alpha\left(\frac{\alpha^{-1}(NB_{n+1}) - \alpha^{-1}(NB_n) - \alpha^{-1}(\dot{1})}{\alpha^{-1}(\dot{4})}\right) \\
&= \alpha\left(\frac{\alpha^{-1}\left(\alpha\left[\alpha^{-1}(NB_{n+1}) - \alpha^{-1}(NB_n) - \alpha^{-1}(\dot{1})\right]\right)}{\alpha^{-1}(\dot{4})}\right) \\
&= \alpha\left(\frac{\alpha^{-1}\left(NB_{n+1} \dot{-} NB_n \dot{-} \dot{1}\right)}{\alpha^{-1}(\dot{4})}\right) = \frac{\dot{B}_{n+1} \dot{-} \dot{B}_n \dot{-} \dot{1}}{\dot{4}}_{\alpha},
\end{aligned}$$

which ends the proof. □

Now, we will give the generating function for the non-Newtonian balancing numbers.

Theorem 7. *Let $z \in \mathbb{R}_{\alpha}$. Then, the generating function for the non-Newtonian balancing numbers is given by*

$$g(z) = \frac{z}{\dot{1} \dot{-} \dot{6} \dot{\times} z \dot{+} z^{\dot{2}}}_{\alpha}.$$

Proof. Assume that the generating function of the non-Newtonian balancing sequence $\{NB_n\}$ has the form $g(z) = {}_{\alpha}\sum_{n=0}^{\infty} NB_n \dot{\times} z^{\dot{n}}$. Then, using the recurrence relation for non-Newtonian balancing numbers, we have

$$g(z) = {}_{\alpha}\sum_{n=0}^{\infty} NB_n \dot{\times} z^{\dot{n}} = NB_0 \dot{\times} z^{\dot{0}} \dot{+} NB_1 \dot{\times} z^{\dot{1}} \dot{+} \ldots \dot{+} NB_n \dot{\times} z^{\dot{n}} \dot{+} \ldots$$

$$= \dot{0} \dot{\times} \alpha\left(\left(\alpha^{-1}(z)\right)^{0}\right) \dot{+} \dot{1} \dot{\times} \alpha\left(\left(\alpha^{-1}(z)\right)^{1}\right) \dot{+}_{\alpha}\sum_{n=2}^{\infty} NB_n \dot{\times} z^{\dot{n}}$$

$$= \dot{0} \dot{\times} \alpha(1) \dot{+} \dot{1} \dot{\times} \alpha\left(\alpha^{-1}(z)\right) \dot{+}_{\alpha}\sum_{n=2}^{\infty} \left(\dot{6} \dot{\times} NB_{n-1} \dot{-} NB_{n-2}\right) \dot{\times} z^{\dot{n}}$$

$$= \alpha\left(\alpha^{-1}(\dot{0}) \cdot \alpha^{-1}(\alpha(1))\right) \dot{+} \alpha\left(\alpha^{-1}(\dot{1}) \cdot \alpha^{-1}(z)\right)$$

$$\dot{+}_{\alpha}\sum_{n=2}^{\infty} \dot{6} \dot{\times} NB_{n-1} \dot{\times} z^{\dot{n}} \dot{-}_{\alpha}\sum_{n=2}^{\infty} NB_{n-2} \dot{\times} z^{\dot{n}}$$

$$= \alpha(0 \cdot 1) \dot{+} \alpha\left(1 \cdot \alpha^{-1}(z)\right)$$

$$\dot{+} \dot{6} \dot{\times} z \dot{\times}_{\alpha}\sum_{n=2}^{\infty} NB_{n-1} \dot{\times} z^{(n-1)^{\cdot}} \dot{-} z^{2} \dot{\times}_{\alpha}\sum_{n=2}^{\infty} NB_{n-2} \dot{\times} z^{(n-2)^{\cdot}}$$

$$= \alpha(0) \dot{+} z \dot{+} \dot{6} \dot{\times} z \dot{\times}_{\alpha}\sum_{n=2}^{\infty} NB_{n-1} \dot{\times} z^{(n-1)^{\cdot}} \dot{+} \dot{6} \dot{\times} z \dot{\times} NB_0 \dot{\times} z^{\dot{0}}$$

$$\dot{-} \dot{6} \dot{\times} z \dot{\times} NB_0 \dot{\times} z^{\dot{0}} \dot{-} z^{2} \dot{\times}_{\alpha}\sum_{n=0}^{\infty} NB_n \dot{\times} z^{\dot{n}}$$

$$= \dot{0} \dot{+} z \dot{+} \dot{6} \dot{\times} z \dot{\times}_{\alpha}\sum_{n=0}^{\infty} NB_n \dot{\times} z^{\dot{n}} \dot{-} \dot{0} \dot{-} z^{2} \dot{\times}_{\alpha}\sum_{n=0}^{\infty} NB_n \dot{\times} z^{\dot{n}}$$

$$= z \dot{+} \dot{6} \dot{\times} z \dot{\times} g(z) \dot{-} z^{2} \dot{\times} g(z),$$

since

$$\dot{6} \dot{\times} z \dot{\times} NB_0 \dot{\times} z^{\dot{0}} = \alpha\left(\alpha^{-1}(\dot{6}) \cdot \alpha^{-1}(z) \cdot \alpha^{-1}(\alpha(0))\right)$$
$$= \alpha\left(\alpha^{-1}(\dot{6}) \cdot \alpha^{-1}(z) \cdot 0\right) = \alpha(0) = \dot{0}.$$

Moreover,

$$\dot{1} \dot{\times} g(z) = \alpha\left(\alpha^{-1}(\dot{1}) \cdot \alpha^{-1}(g(z))\right) = \alpha\left(1 \cdot \alpha^{-1}(g(z))\right) = g(z),$$

hence we have

$$\dot{1} \dot{\times} g(z) = z \dot{+} \dot{6} \dot{\times} z \dot{\times} g(z) \dot{-} z^{2} \dot{\times} g(z)$$

and finally

$$g(z) = \frac{z}{\dot{1} \dot{-} \dot{6} \dot{\times} z \dot{+} z^{2}}\alpha,$$

which ends the proof. □

Concluding remarks. As is known, the identity function generates classical arithmetic and the exponential function generates geometric arithmetic. If we choose the identity function I instead of α in the definition of non-Newtonian balancing numbers, then we obtain the classical balancing numbers and the known results of these numbers. If α is the exponential function, then α generates the geometric arithmetic:

geometric addition $x \dot{+} y = e^{\ln x + \ln y} = x \cdot y$,
geometric subtraction $x \dot{-} y = e^{\ln x - \ln y} = \frac{x}{y}$,
geometric multiplication $x \dot{\times} y = e^{\ln x \cdot \ln y} = x^{\ln y} = y^{\ln x}$,
geometric division $x \dot{/} y = e^{\ln x : \ln y} = x^{\frac{1}{\ln y}}, \ y \neq 1$.

In Table 3 we can find balancing type numbers related to geometric arithmetic.

n	0	1	2	3	4	5	6	7
e^{B_n}	e^0	e^1	e^6	e^{35}	e^{204}	e^{1189}	e^{6930}	e^{40391}
e^{C_n}	e^1	e^3	e^{17}	e^{99}	e^{577}	e^{3363}	e^{19601}	e^{114243}
e^{b_n}	e^0	e^0	e^2	e^{14}	e^{84}	e^{492}	e^{2870}	e^{16730}
e^{c_n}	e^{-1}	e^1	e^7	e^{41}	e^{239}	e^{1393}	e^{8119}	e^{47321}

TABLE 3. The balancing type numbers related to geometric arithmetic.

In future work, researchers can investigate the geometric version of the properties of balancing numbers. It will be interesting to continue this research by examining other balancing type numbers in terms of non-Newtonian calculus.

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Anetta Szynal-Liana
The Faculty of Mathematics and Applied Physics
Rzeszów University of Technology
al. Powstańców Warszawy 12
35-959 Rzeszów
Poland
e-mail: aszynal@prz.edu.pl
ORCID: 0000-0001-5508-0640

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