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A new q-Laplace transform with many examples

ABSTRACT. In the spirit of Hahn 1949, the purpose of this paper is to introduce a new q-Laplace transform for a Jackson q-integral $\int_0^a f(t,q)\,d_q(t)$, with upper integration boundary $\frac{1}{s(1-q)}$. For this purpose we redefine this q-integral with a σ-algebra and a discrete measure supported at the points $x = aq^n$, $n \in \mathbb{N}$. Then we prove q-analogues of many well-known Laplace transform formulas, including the formula for the transform of the delta distribution. The paper concludes with a list of q-Laplace transforms for (multiple) q-hypergeometric series, some with function arguments in the first q-real numbers $\mathbb{R}_{\oplus_q}$. Elsewhere, other q-real numbers are defined in similar style as function arguments in formal power series.

CONTENTS

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1. Introduction

In q-difference equations, the derivatives of differential equations are replaced by q-derivatives. The theory of q-difference equations is not fully explored, and there is, in particular, a need to define a correct q-Laplace transform. Obviously, when this is known, all q-difference equations, which are q-analogues of the corresponding ordinary, homogeneous differential equations with constant coefficients, can be solved. The reason is that these differential equations have exponential and/or trigonometric solutions and our q-analogues of these two functions (and of the derivative as well) have the same q-Laplace transform. As a guide to the reader, who is assumed to have some knowledge about Laplace transforms, all the details will be explained systematically. Hint: All beginner's books on the Laplace transform give a series of formulas, which are often repeated in each book. The weak point in q-calculus is that there is no q-analogue of generalized integrals, although these are stated e.g. in the book by Gasper and Rahman [13]. This means that all formulas with generalized integrals for Laplace transforms, which change order of integration or which make a linear substitution in integrals, have no q-analogue. The experienced reader may now already guess which formulas for Laplace transforms that we cannot q-deform. We will, however, q-deform all the other, more transparent, ones.

The paper is organized as follows: In Section 1 we make a brief introduction to the subject. Section 2 introduces the notations, some of which can be found in our book [8]. In Section 3 we briefly repeat the first q-real number $\mathbb{R}_{\oplus_q}$ from [10]. Section 4 presents a corrected version of the q-Laplace transform by Chung, Kim and Kwon [2], and in Subsection 4.1 the fundamental prerequisites for the Jackson q-integral are summarized. In Subsection 4.2, finally, we give a correct version of the q-Laplace transform. In Subsection 4.3 we only present some typical proofs of several q-hypergeometric q-Laplace transforms.

2. q-Calculus definitions

We now repeat some notations from [8]. Throughout, $\equiv$ denotes a definition and $\cong$ denotes a formal equality.

Definition 1. Let $\delta > 0$ be an arbitrary small number. We will always use the following branch of the logarithm: $-\pi + \delta < \mathrm{Im}(\log q) \leq \pi + \delta$. This defines a simply connected domain in the complex plane.

The power function is defined by

$$q^a \equiv e^{a \log(q)}.$$

The following notation is often used when long exponents appear.

$$\mathrm{QE}(x) \equiv q^x.$$

Definition 2 ([8, p. 19]). The q-analogues of a complex number a, a natural number n and the factorial are defined as follows:

$$\{a\}_q \equiv \frac{1-q^a}{1-q},\ q \in \mathbb{C}\backslash\{0,1\},$$

$$\{n\}_q \equiv \sum_{k=1}^{n} q^{k-1},\ \{0\}_q = 0,\ q \in \mathbb{C}\backslash\{0,1\},$$

$$\{n\}_q! \equiv \prod_{k=1}^{n} \{k\}_q,\ \{0\}_q! \equiv 1,\ q \in \mathbb{C}\backslash\{0,1\}.$$

Definition 3. The q-shifted factorial [8] is defined by

$$\langle a;q\rangle_n \equiv \prod_{m=0}^{n-1}(1-q^{a+m}).$$

Sometimes we also use

$$(a;q)_n \equiv \prod_{m=0}^{n-1}(1-aq^m).$$

Definition 4. In the following, $\frac{\mathbb{C}}{\mathbb{Z}}$ will denote the space of complex numbers mod$\frac{2\pi i}{\log q}$. This is isomorphic to the cylinder $\mathbb{R} \times e^{2\pi i\theta},\ \theta \in \mathbb{R}$. The operator

$$\widetilde{}: \frac{\mathbb{C}}{\mathbb{Z}} \to \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by the 2-torsion

$$a \mapsto a + \frac{\pi i}{\log q}. \tag{1}$$

By (1) it follows that

$$\widetilde{\langle a;q\rangle}_n = \prod_{m=0}^{n-1}(1+q^{a+m}),$$

where this time the tilde denotes the involution which changes a minus sign to a plus sign in all the n factors of $\langle a;q\rangle_n$.

For relatively prime m, l, the generalized tilde operator

$$\widetilde{\tfrac{m}{l}}: \frac{\mathbb{C}}{\mathbb{Z}} \to \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by

$$a \mapsto a + \frac{2\pi i m}{l \log q}. \tag{2}$$

We also need another generalization of the tilde operator.

$$
{}_k\widetilde{\langle a;q\rangle}_n \equiv \prod_{m=0}^{n-1}\left(\sum_{i=0}^{k-1} q^{i(a+m)}\right). \tag{3}
$$

Formula (3) is used in (4).

The following simple congruence rules [8] follow from (2).

Theorem 1.

$$
\widetilde{{}^{\frac{m}{l}}a} \pm b \equiv \widetilde{{}^{\frac{m}{l}}(a\pm b)} \ \left(\mathrm{mod}\ \frac{2\pi i}{\log q}\right),
$$

$$
\sum_{k=1}^{n} \widetilde{{}^{\frac{1}{n}} \pm a_k} \equiv \sum_{k=1}^{n} \pm a_k \ \left(\mathrm{mod}\ \frac{2\pi i}{\log q}\right),
$$

$$
\frac{m}{l} \times \widetilde{a} \equiv \widetilde{{}^{\frac{m}{2l}}\frac{am}{l}} \ \left(\mathrm{mod}\ \frac{2\pi i}{\log q}\right),
$$

$$
\mathrm{QE}(\widetilde{{}^{\frac{m}{l}}a}) = \mathrm{QE}(a)e^{\frac{2\pi i m}{l}},
$$

where the second equation is a consequence of the fact that we work $\mathrm{mod}\frac{2\pi i}{\log q}$.

Definition 5.

$$
\langle\lambda;q\rangle_{kn} \equiv \langle\triangle(q;k;\lambda);q\rangle_n \equiv \prod_{m=0}^{k-1}\left\langle\frac{\lambda+m}{k};q\right\rangle_n \times_k \left\langle\frac{\widetilde{\lambda+m}}{k};q\right\rangle_n. \tag{4}
$$

We also use the notation $\triangle(q;k;\lambda)$ as a parameter in q-hypergeometric functions.

If λ is a vector, we mean the corresponding product of vector components. If λ is replaced by a sequence of numbers, separated by commas, we mean the corresponding product, as in the case of q-factorials.

The last factor in (4) corresponds to k^{nk}.

Definition 6 ([8, (1.45)]). The Γ_q function is defined by

$$
\Gamma_q(z) \equiv \begin{cases} \dfrac{\langle 1;q\rangle_\infty}{\langle z;q\rangle_\infty}(1-q)^{1-z}, & \text{if } 0<|q|<1; \\[2ex] \dfrac{\langle 1;q^{-1}\rangle_\infty}{\langle z;q^{-1}\rangle_\infty}(q-1)^{1-z}q^{\binom{z}{2}}, & \text{if } |q|>1. \end{cases}
$$

Definition 7 ([8, (1.49)]). Let S_r denote the additional poles of Γ_q, vertical if q is real and slanting if q is complex. Then the generalized Γ_q function, a function $(\mathbb{C}\setminus(\{\mathbb{Z}\leq 0\}\cup S_r))^{p+r}\times\mathbb{C}\mapsto\mathbb{C}$, is defined as follows:

$$\Gamma_q\left[\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_r \end{array}\right]\equiv\frac{\Gamma_q(a_1)\ldots\Gamma_q(a_p)}{\Gamma_q(b_1)\ldots\Gamma_q(b_r)}.$$

This is a modest attempt to present a new notation for q-calculus and in particular for q-hypergeometric series, which is compatible with the old notation. With this notation, q-hypergeometric function- and hypergeometric function equations become very similar.

Definition 8. Generalizing Heine series, we shall define a q-hypergeometric series by

$$\begin{aligned}&{}_{p+p'}\phi_{r+r'}\left[\begin{array}{c}\hat{a}_1,\ldots,\hat{a}_p \\ \hat{b}_1,\ldots,\hat{b}_r\end{array}\middle|q;z\middle\|\begin{array}{c}\prod_i f_i(k)\\ \prod_j g_j(k)\end{array}\right]\\ &\equiv\sum_{k=0}^{\infty}\frac{\langle\hat{a}_1;q\rangle_k\ldots\langle\hat{a}_p;q\rangle_k}{\langle 1,\hat{b}_1;q\rangle_k\ldots\langle\hat{b}_r;q\rangle_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+r+r'-p-p'}z^k\frac{\prod_i f_i(k)}{\prod_j g_j(k)},\end{aligned}$$

where

$$\widehat{a}\equiv a\vee\widetilde{a}\vee\widetilde{\tfrac{m}{l}a}\vee_k\widetilde{a}\vee\triangle(q;l;\lambda).$$

In case of $\triangle(q;l;\lambda)$ the index is adjusted accordingly. It is assumed that the denominator contains no zero factors, i.e. $\widehat{b_k}\neq -l+\frac{2m\pi i}{\log q}$, $k=1,\ldots,r$, $l,m\in\mathbb{N}$ [18]. We assume that the $f_i(k)$ and $g_j(k)$ contain p' and r' factors of the form $\langle\widehat{a(k)};q\rangle_k$ or $(s(k);q)_k$ respectively.

The following definition, as in the one-variable case, allows easy limits for parameters to $\pm\infty$.

The first definition is a q-analogue of [19, (24), p. 38], in the spirit of Srivastava. The second definition is a q-analogue of [19, (24), p. 38] with the restraint [19, (29), p. 38], due to Karlsson. It will be clear from the context which of the definitions we use.

Definition 9 ([8, p. 367 f]). Let the vectors

$$(a),(b),(g_i),(h_i),(a'),(b'),(g_i'),(h_i')$$

have lengths

$$A,B,G_i,H_i,A',B',G_i',H_i'.$$

Let

$$1+B+B'+H_i+H_i'-A-A'-G_i-G_i'\geq 0,\ i=1,\ldots,n.$$

Then the generalized q-Kampé de Fériet function is defined by

$$\Phi^{A+A':G_1+G'_1;\ldots;G_n+G'_n}_{B+B':H_1+H'_1;\ldots;H_n+H'_n}\left[\begin{array}{c}(\hat{a}):(\hat{g_1});\ldots;(\hat{g_n})\\(\hat{b}):(\hat{h_1});\ldots;(\hat{h_n})\end{array}\middle|\vec{q};\vec{x}\middle\|\begin{array}{c}(a'):(g'_1);\ldots;(g'_n)\\(b'):(h'_1);\ldots;(h'_n)\end{array}\right]$$
$$\equiv\sum_{\vec{m}}\frac{\langle(\hat{a});q_0\rangle_m(a')(q_0,m)\prod_{j=1}^n(\langle(\hat{g_j});q_j\rangle_{m_j}((g'_j)(q_j,m_j)x_j^{m_j})}{\langle(\hat{b});q_0\rangle_m(b')(q_0,m)\prod_{j=1}^n(\langle(\hat{h_j});q_j\rangle_{m_j}(h'_j)(q_j,m_j)\langle 1;q_j\rangle_{m_j})}$$
$$\times(-1)^{\sum_{j=1}^n m_j(1+H_j+H'_j-G_j-G'_j+B+B'-A-A')}$$
$$\times\mathrm{QE}\left((B+B'-A-A')\binom{m}{2},q_0\right)$$
$$\times\prod_{j=1}^n\mathrm{QE}\left((1+H_j+H'_j-G_j-G'_j)\binom{m_j}{2},q_j\right).$$

It is assumed that there are no zero factors in the denominator and that $(a')(q_0,m),(g'_j)(q_j,m_j),(b')(q_0,m),(h'_j)(q_j,m_j)$ contain factors of the form $\langle a(\hat{k});q\rangle_k$, $(s;q)_k,(s(k);q)_k$ or $\mathrm{QE}\,(f(\vec{m}))$.

Definition 10 ([8, p. 368 f]). Let the vectors

$$(a),(b),(g_i),(h_i),(a'),(b'),(g'_i),(h'_i)$$

have lengths

$$A,B,G,H,A',B',G',H'.$$

Let

$$1+B+B'+H+H'-A-A'-G-G'\geq 0.$$

Then the generalized q-Kampé de Fériet function is defined by

$$\Phi^{A+A':G+G'}_{B+B':H+H'}\left[\begin{array}{c}(\hat{a}):(\hat{g_1});\ldots;(\hat{g_n})\\(\hat{b}):(\hat{h_1});\ldots;(\hat{h_n})\end{array}\middle|\vec{q};\vec{x}\middle\|\begin{array}{c}(a'):(g'_1);\ldots;(g'_n)\\(b'):(h'_1);\ldots;(h'_n)\end{array}\right]$$
$$\equiv\sum_{\vec{m}}\frac{\langle(\hat{a});q_0\rangle_m(a')(q_0,m)\prod_{j=1}^n(\langle(\hat{g_j});q_j\rangle_{m_j}((g'_j)(q_j,m_j)x_j^{m_j})}{\langle(\hat{b});q_0\rangle_m(b')(q_0,m)\prod_{j=1}^n(\langle(\hat{h_j});q_j\rangle_{m_j}(h'_j)(q_j,m_j)\langle 1;q_j\rangle_{m_j})}$$
$$\times(-1)^{\sum_{j=1}^n m_j(1+H+H'-G-G'+B+B'-A-A')}$$
$$\times\mathrm{QE}\left((B+B'-A-A')\binom{m}{2},q_0\right)$$
$$\times\prod_{j=1}^n\mathrm{QE}\left((1+H+H'-G-G')\binom{m_j}{2},q_j\right),$$

where

$$\widehat{a}\equiv a\vee\widetilde{a}\vee\widetilde{\tfrac{m}{l}a}\vee_k\widetilde{a}\vee\triangle(q;l;\lambda).$$

It is assumed that there are no zero factors in the denominator. We assume that $(a')(q_0,m),(g'_j)(q_j,m_j),(b')(q_0,m),(h'_j)(q_j,m_j)$ contain factors of the form $\langle a(\hat{k});q\rangle_k$, $(s;q)_k,(s(k);q)_k$ or $\mathrm{QE}\,(f(\vec{m}))$.

The numbers in front of the colon represent the number of q-shifted factorials with index m in numerator and the denominator. The numbers after the colon denote the number of q-shifted factorials with index m_i in numerator and the denominator. Equally, the numbers after semicolon denote the number of q-shifted factorials with index m_i in numerator and denominator. We can leave out G_2 if it is equal to G_1 for two variables etc. Every ∞ corresponds to multiplication with 1.

Definition 11. The q-derivative is defined by

$$(\mathrm{D}_q\varphi)(x) \equiv \begin{cases} \dfrac{\varphi(x)-\varphi(qx)}{(1-q)x}, & \text{when } q\in\mathbb{C}\backslash\{1\},\ x\neq 0;\\ \dfrac{d\varphi}{dx}(x), & \text{when } q=1;\\ \dfrac{d\varphi}{dx}(0), & \text{when } x=0.\end{cases}$$

Definition 12. Let the Gaussian q-binomial coefficients be defined by

$$\binom{n}{k}_q \equiv \frac{\langle 1;q\rangle_n}{\langle 1;q\rangle_k\langle 1;q\rangle_{n-k}}, k=0,1,\ldots,n.$$

Theorem 2. *The q-binomial theorem:*

$$\sum_{n=0}^{\infty}\frac{\langle a;q\rangle_n}{\langle 1;q\rangle_n}z^n = \frac{(zq^a;q)_\infty}{(z;q)_\infty},$$

$|z|<1,\ 0<|q|<1.$

Definition 13. If $|q|>1 \vee 0<|q|<1, |z|<|1-q|^{-1}$, the q-exponential function $\mathrm{E}_q(z)$ is defined by

$$\mathrm{E}_q(z) \equiv \sum_{k=0}^{\infty}\frac{1}{\{k\}_q!}z^k. \tag{5}$$

By the Euler equation (6), the meromorphic continuation of $\mathrm{E}_q(z)$ is given by

$$\frac{1}{(z(1-q);q)_\infty}.$$

Thus the meromorphic function $\frac{1}{(z(1-q);q)_\infty}$, with simple poles at $\frac{q^{-k}}{1-q}$, $k\in\mathbb{N}$ is a good substitute for $\mathrm{E}_q(z)$ in the whole complex plane. We shall however continue to designate this function $\mathrm{E}_q(z)$, since it plays an important role in the operator theory.

The q-difference for $\mathrm{E}_q(z)$ is

$$\mathrm{D}_q\ \mathrm{E}_q(az) = a\ \mathrm{E}_q(az).$$

There is another q-exponential function which is entire when $0 < |q| < 1$ and which converges when $|z| < |1-q|^{-1}$ if $|q| > 1$. To obtain it, the base in (5) must be inverted, i.e. $q \to \frac{1}{q}$. This is a common theme in q-calculus.

Definition 14.

$$\mathrm{E}_{\frac{1}{q}}(z) \equiv \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}}{\{k\}_q!} z^k.$$

We immediately obtain

$$\mathrm{E}_{\frac{1}{q}}(z) = \prod_{n=0}^{\infty} \left(1 + (1-q)zq^n\right),\ 0 < |q| < 1.$$

The q-difference equation for $\mathrm{E}_{\frac{1}{q}}(z)$ is

$$\mathrm{D}_q \mathrm{E}_{\frac{1}{q}}(az) = a\mathrm{E}_{\frac{1}{q}}(qaz),$$

which reduces to the differential equation of the exponential function when q tends to unity.

For later use, we shall need a third q-exponential function:

Definition 15.

$$\widetilde{\mathrm{E}_{\frac{1}{q}}}(z) \equiv \sum_{k=0}^{\infty} \frac{(k+1)q^{\binom{k+1}{2}}}{\{k+1\}_q!} z^k.$$

Definition 16. Euler found the following two extra q-analogues of the exponential function:

$$\begin{aligned} \mathrm{e}_q(z) &\equiv\ {}_1\phi_0(\infty; -|q; z) \\ &\equiv \sum_{n=0}^{\infty} \frac{z^n}{\langle 1; q\rangle_n} = \frac{1}{(z;q)_\infty},\ |z| < 1,\ 0 < |q| < 1. \end{aligned} \tag{6}$$

$$\mathrm{e}_{\frac{1}{q}}(z) \equiv\ {}_0\phi_0(-; -|q; -z) \equiv \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{\langle 1; q\rangle_n} z^n = (-z;q)_\infty,\ 0 < |q| < 1.$$

The second function is an entire function just as the usual exponential function.

Definition 17. We can now define four q-analogues of the trigonometric functions. In the first two equations, $|q| > 1$, or $0 < |q| < 1$ and $|x| < |1-q|^{-1}$.

$$\mathrm{Sin}_q(x) \equiv \frac{1}{2}(\mathrm{E}_q(ix) - \mathrm{E}_q(-ix)).$$

$$\mathrm{Cos}_q(x) \equiv \frac{1}{2}(\mathrm{E}_q(ix) + \mathrm{E}_q(-ix)).$$

$$\mathrm{Sin}_{\frac{1}{q}}(x) \equiv \frac{1}{2}(\mathrm{E}_{\frac{1}{q}}(ix) - \mathrm{E}_{\frac{1}{q}}(-ix)).$$

$$\mathrm{Cos}_{\frac{1}{q}}(x) \equiv \frac{1}{2}(\mathrm{E}_{\frac{1}{q}}(ix) + \mathrm{E}_{\frac{1}{q}}(-ix)),$$

where $x \in \mathbb{C}$ in the last two equations.

Definition 18 ([8]). Three q-Appell function are defined by [9]:

$$\Phi_1(a; b, b'; c|q; x_1, x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a; q\rangle_{m_1+m_2} \langle b; q\rangle_{m_1} \langle b'; q\rangle_{m_2}}{\langle 1; q\rangle_{m_1} \langle 1; q\rangle_{m_2} \langle c; q\rangle_{m_1+m_2}} x_1^{m_1} x_2^{m_2},$$

$\max(|x_1|, |x_2|) < 1.$

$$\Phi_2(a; b, b'; c, c'|q; x_1, x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a; q\rangle_{m_1+m_2} \langle b; q\rangle_{m_1} \langle b'; q\rangle_{m_2}}{\langle 1; q\rangle_{m_1} \langle 1; q\rangle_{m_2} \langle c; q\rangle_{m_1} \langle c'; q\rangle_{m_2}} x_1^{m_1} x_2^{m_2},$$

$|x_1| \oplus_q |x_2| < 1.$

$$\Phi_4(a; b; c, c'|q; x_1, x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a; q\rangle_{m_1+m_2} \langle b; q\rangle_{m_1+m_2}}{\langle 1; q\rangle_{m_1} \langle 1; q\rangle_{m_2} \langle c; q\rangle_{m_1} \langle c'; q\rangle_{m_2}} x_1^{m_1} x_2^{m_2},$$

$|\sqrt{x_1}| \oplus_q |\sqrt{x_2}| < 1.$

Remark 1. The function Φ_1 occurs in formulas (33), (36). The function Φ_2 occurs in formulas (31), (43). The function Φ_4 occurs in (30).

Since the number of q-shifted factorials in denominators is larger than in numerators for q-confluent functions, by the quotient criterion, the convergence regions are drastically increased. These convergence regions in the confluent hypergeometric case were only given by Srivastava and Karlsson in [19].

Definition 19.

$$\Psi_1(a; b; c, c'|q; x_1, x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a; q\rangle_{m_1+m_2} \langle b; q\rangle_{m_1}}{\langle 1; q\rangle_{m_1} \langle 1; q\rangle_{m_2} \langle c; q\rangle_{m_1} \langle c'; q\rangle_{m_2}} x_1^{m_1} x_2^{m_2},$$

$|x_1| < 1,\ |(1-q)x_2| < \infty,$

$$\Psi_2(a; c, c'|q; x_1, x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a; q\rangle_{m_1+m_2}}{\langle 1; q\rangle_{m_1} \langle 1; q\rangle_{m_2} \langle c; q\rangle_{m_1} \langle c'; q\rangle_{m_2}} x_1^{m_1} x_2^{m_2},$$

$|(1-q)x_1| < \infty,\ |(1-q)x_2| < \infty,$

$$\Upsilon_1(a; b; c|q; x_1, x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a; q\rangle_{m_1+m_2} \langle b; q\rangle_{m_1}}{\langle 1; q\rangle_{m_1} \langle 1; q\rangle_{m_2} \langle c; q\rangle_{m_1+m_2}} x_1^{m_1} x_2^{m_2},$$

$|x_1| < 1,\ |(1-q)x_2| < \infty.$

$$\Upsilon_2(a,a';c|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1}\langle a';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}} x_1^{m_1}x_2^{m_2},$$

$|(1-q)x_1| < \infty,\ |(1-q)x_2| < \infty.$

$$\Upsilon_3(a;c|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}} x_1^{m_1}x_2^{m_2},$$

$|(1-q)x_1| < \infty,\ |(1-q)^2x_2| < \infty.$

$$\Xi_1(a,a';b;c|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1}\langle a';q\rangle_{m_2}\langle b;q\rangle_{m_1}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}} x_1^{m_1}x_2^{m_2},$$

$|x_1| < 1,\ |(1-q)x_2| < \infty.$

$$\Xi_2(a;b;c|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1}\langle b;q\rangle_{m_1}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}} x_1^{m_1}x_2^{m_2},$$

$|x_1| < 1,\ |(1-q)^2x_2| < \infty.$

3. Survey of q-real numbers

The q-real numbers give a convenient notation for q-additions in formal power series, in particular for q-exponential and q-trigonometric functions. There is a one-to-one correspondence between the convergence regions of the two q-Lauricella functions $\Phi_{\mathrm{A}}^{(n)}$ and $\Phi_{\mathrm{C}}^{(n)}$, and the existence of q-real numbers with n letters (or variables).

Definition 20 ([8, p. 24]). Let $a, b \in \mathbb{R}$. Then the NWA q-addition is given by

$$(a \oplus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k},\ n = 0,1,2,\ldots,\ a \oplus_q b \in \mathbb{R}_{\oplus_q}. \tag{7}$$

In particular, $(a \oplus_q b)^0 \equiv 1$. Furthermore, we put

$$(a \ominus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k (-b)^{n-k},\ n = 0,1,2,\ldots.$$

Definition 21 ([10]). Let $I^n \subset \mathbb{R}^n, I \equiv (0,1]$ denote the half-open n-dimensional hypercube. For q fixed, the q-real numbers $\mathbb{R}_{\oplus_q}$ form a subset of the disjoint union of all hypercubes

$$\mathbb{R}_{\oplus_q} \subset \bigcup_{n=2}^{\infty} I^n.$$

For the following definition one could compare with the formula [8, 4.74 p. 110]:

Definition 22. Given $k \in \mathbb{N}$, the formula

$$m_0 + m_1 + \cdots + m_j = k$$

determines a set $J_{m_0,\ldots,m_j} \in \mathbb{N}^{j+1}$.

Definition 23. For $\vec{m} \in \mathbb{N}^n$ put

$$|\vec{m}| \equiv m_1 + \ldots + m_n.$$

If $f(x)$ is the formal power series $\sum_{l=0}^{\infty} a_l x^l$, its kth NWA-power is given by

$$(\oplus_{q,l=0}^{\infty} a_l x^l)^k \equiv (a_0 \oplus_q a_1 x \oplus_q \ldots)^k \equiv \sum_{|\vec{m}|=k} \prod_{m_l \in J_{m_0,\ldots,m_j}} (a_l x^l)^{m_l} \binom{k}{\vec{m}}_q.$$

For $a = (a_1, \ldots, a_n) \in I^n$ put

$$(a_1 \oplus_q a_2 \oplus_q \ldots \oplus_q a_n)^k \equiv \sum_{\vec{|m|}=k} \prod_{m_l \in J_{m_0,\ldots,m_j}} (a_l)^{m_l} \binom{k}{\vec{m}}_q.$$

Conjecture 1 ([10]). If the function

$$F(k) \equiv (a_1 \oplus_q a_2 \oplus_q \ldots \oplus_q a_n)^k$$

has exactly one absolute maximum in $\mathbb{N}$, then we have $\lim_{k \to \infty} F(k) = 0$.

Definition 24. We have $\vec{a} := (a_1, \ldots, a_n) \in \mathbb{R}_{\oplus_q}$ exactly when the function $F(k)$ has exactly one absolute maximum.

For the commutative monoid $\mathbb{R}_{\oplus_q}$ we note the following definitions and formulas:

Definition 25. Assume that $\sim$ means equality on $\mathbb{R}[[x]]$ [8, p. 101].

There is a certain linear functional $v: \mathbb{R}[[x]] \times \mathbb{R}_q \mapsto \mathbb{R}$, with $v(f, 0) = a_0 \in \mathbb{R}$, called the evaluation.

Theorem 3. *The q-addition* (7) *has the following properties, for* $\alpha, \beta, \gamma \in \mathbb{R}_{\oplus_q}$*:*
Commutativity:

$$\alpha \oplus_q \beta \sim \beta \oplus_q \alpha.$$

Associativity

$$(\alpha \oplus_q \beta) \oplus_q \gamma \sim \alpha \oplus_q (\beta \oplus_q \gamma).$$

To be able to formulate equation (22), q-Laplace transform of multiplication with $\mathrm{E}_q(\alpha t)$, we introduce the following extension of the umbral calculus. Compare with the three formulas in [8, p. 103].

Definition 26. The q-addition $x \boxplus_q y$ is defined by another q-Taylor formula:

$$F(x \boxplus_q y) \equiv \sum_{n=0}^{\infty} \frac{y^n}{\{n\}_q!} q^{\binom{n}{2}} \mathrm{D}_{q,x}^n F(q^{-n}x). \tag{8}$$

As before, only positive integer powers of $(x \boxplus_q y)$ are used. We call the function argument in (8) $\mathbb{R}_{\boxplus_q}$.

4. On the q-Laplace transform

Several authors have tried to introduce different q-Laplace transforms with upper q-integration limits ∞ and $(s(1-q))^{-1}$. The latter converges to ∞ when $q \to 1^-$. Also a time-scale approach to this problem has been published by Martin Bohner et al. [1], who used operators similar to our q-real numbers.

This paper was enabled by Erik Koelink and Tom Koornwinder [16], who, in 1992 presented the correct Γ_q function expression as a q-integral with $\mathrm{E}_{\frac{1}{q}}(x)$, which enables the correct q-integration by parts proofs. This was possible by using the product expansion for the q-exponential $\mathrm{E}_{\frac{1}{q}}(x)$.

Later, in 2005, Kac et al. [3] payed attention to this, and showed how to express the Γ_q function as a q-integral with $\mathrm{E}_q(x)$ times an extra factor, a q-analogue of 1. Finally, in 2014, Chung, Kim and Kwon [2] tried to find a q-Laplace transform, which would be useful for practical purposes. We shall now find q-analogues of many properties of the Laplace transform by improving the treatment of the cited paper [2], so as to obtain formulas that do not involve q-integrals of the form $\int_0^\infty$, which may be difficult to define. We point out that we shall use the Hahn q-Laplace transform [15], correcting a slight misprint.

4.1. Preliminaries: properties of the q-integral.

We first repeat the definitions of q-integrals from [8]. Note that the definitions of these q-integrals for $a, b \in \mathbb{R}_{\oplus_q}$ in [8, (4.80)] are only for umbral use.

Definition 27. The Jackson q-integral is defined by

$$\int_a^b f(t,q)\, d_q(t) \equiv \int_0^b f(t,q)\, d_q(t) - \int_0^a f(t,q)\, d_q(t),\ \ a,b \in \mathbb{R},$$

where

$$\int_0^a f(t,q)\, d_q(t) \equiv a(1-q) \sum_{n=0}^{\infty} f(aq^n, q) q^n,\ \ 0 < |q| < 1,\ \ a \in \mathbb{R}. \tag{9}$$

We now show how the main integral theorems are included in q-analysis.

Definition 28. Let $E_n \equiv \{aq^n\},\ n = 1,2,3,\ldots,\ a \in \mathbb{R}^\star$ be distinct singleton sets, with respective measures $a(1-q)q^n$. Then the σ–algebra is defined by $\mathcal{M} \equiv \{\mu(E_n) = (1-q)q^n\}_{n=0}^{\infty}$.

By the measure definition, we then have

$$\mu\left(\bigcup_{0}^{\infty} E_n\right) = \sum_{n=0}^{\infty} \mu(E_n).$$

The q-integral (9) can be written in the form

$$\sum_{n=0}^{\infty} f(aq^n)\mu(E_n),$$

which is the q-integral of f with respect to an infinite discrete measure, that converges weakly to a Lebesgue measure as $q \to 1^-$.

Put

$$F(x) \equiv \int_0^x f(t,q)\, d_q(t).$$

Then [5]:

(1) If $F(x)$ is well-defined, then $\mathrm{D}_q f(x) = F(x)$.
(2) If $f(x)$ is continuous on the closed disk $\mathrm{D}(0, r^+)$, then $F(x)$ is well-defined for any $x \in \mathrm{D}(0, r^+)$. In fact there exists $K > 0$ such that $|f(q^n x)q^n| \leq K|q|^n$, which guarantees the convergence of the infinite sum.

Let $\mathcal{L}_q^1$ denote the Banach space of all q-integrable functions on the interval I. The following three theorems are proved analogously to the standard case.

Theorem 4 ([12, p. 52]). *Triangle inequality.*
Let $f(x) \in \mathcal{L}_q^1$. Then

$$\left|\int f d\mu\right| \leq \int |f|\, d\mu.$$

Theorem 5. *Let $\{f_n\}_0^\infty$ be a continuous sequence with limit function*

$$f = \lim_{n\to\infty} f_n, \tag{10}$$

which converges uniformly. Then we have

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu. \tag{11}$$

Proof.

$$\left|\int f_n d\mu - \int f d\mu\right| \leq \int |fn - f| d\mu \leq \int \|f_n - f\| d\mu$$
$$= \|f_n - f\| \int d\mu \to 0,\ n \to \infty.$$ □

Theorem 6. *If the conditions in* (10) *are satisfied, then we have*

$$\int \sum_{n=0}^{\infty} f_n(x)d\mu = \sum_{n=0}^{\infty} \int f_n(x)d\mu.$$

Proof. Use (11). □

Definition 29. Let Λ_q be the set of all piecewise q-differentiable functions on $(0, \infty)$. We only consider functions in $\mathbb{C}[[z]]$, except the Heaviside step and Dirac functions, which have known q-Laplace transforms.

Definition 30. The equivalence relation $\mathcal{A}_q$ on $\mathbb{R}^+$ is defined as follows: Elements $a, b > 0$ belong to the same equivalence class,

$$a \sim b \Longleftrightarrow \exists n \in \mathbb{Z} : \log \frac{a}{b} = n \log q.$$

The equivalence class $[a], a \in \mathbb{R}^+$ is defined as follows:

$$[a] \equiv \{x \mid \exists n \in \mathbb{Z} : x = aq^n\}.$$

Theorem 7 ([8, p. 204], [14], [17]). *Multiplicative substitution in a q-integral:*

$$\int_0^x f(t,q)\ d_q(t) = b \int_0^{\frac{x}{b}} f(bt,q)\ d_q(t). \tag{12}$$

Theorem 8. *Power substitutions* $f(x^k, q) \mapsto f(t, q), k \in \mathbb{N}$ *in q-integrals:*

$$\int_0^a f(x^k,q)\ d_q(x) = \frac{1}{\{k\}_q} \int_0^a t^{\frac{1-k}{k}} f(t,q)\ d_{q^k}(t),\ \ a \in \mathbb{R}.$$

Proof. We compute the right hand side.

$$\begin{aligned} &\frac{1-q}{1-q^k} a(1-q^k) \sum_{n=0}^{\infty} q^{(1-k)n} f(aq^{kn}, q) q^{kn} \\ &= a(1-q) \sum_{n=0}^{\infty} f(aq^{kn}, q) q^n. \end{aligned}$$

This equals the left hand side. □

The formulas for substitution in q-integrals above lead to formulas for re-scaling of the measure $\mu(E_n)$, compare with Diaz, Pariguan [4, p. 3]. Assume that $0 < a < b$, $c > 0$.
For (9) we have :

$$\mu(E_n)[ca, cb] = c\mu(E_n)[a, b].$$

4.2. One-sided q-Laplace transform. This section aims at developing the q-Laplace transform to be used when solving q-difference equations.

We start with the following formula [16]:

$$\Gamma_q(z) = \int_0^{\frac{1}{1-q}} t^{z-1} \mathrm{E}_{\frac{1}{q}}(-qt)\, d_q(t), \ \mathrm{Re}(z) > 0. \tag{13}$$

We shall now present a corrected version of Hahn [15, (9.1), p. 371].

Theorem 9. *Hahn's definition* [15, (9.1), p. 371] *does not converge to the Laplace transform when* $\lim_{q\to 1^-}$.

Proof. Hahn writes in other notation

$$\mathcal{L}_q(s) \equiv \frac{1}{1-q} \int_0^{\frac{1}{s}} f(t) \mathrm{E}_{\frac{1}{q}} \left(\frac{-qst}{1-q} \right) d_q(t)$$
$$\overset{\text{by(12)}}{=} \int_0^{\frac{1}{s(1-q)}} f(t(1-q)) \mathrm{E}_{\frac{1}{q}}(-qst)\, d_q(t),$$

where we put $b = 1 - q$ in (12). But when $\lim_{q\to 1^-}$ the function argument in the q-integral converges to 0, which makes no sense. □

The classical Laplace transform, well known to applied mathematicians and engineers, maps suitable real-valued or complex-valued functions $f(t)$, $t > 0$, to corresponding functions $F(s)$ of another variable s, which are defined for $\mathrm{Re}\, s > s_0$, where s_0 is function-specific.

When defining the q-Laplace transform, we must use formula (13) and the second q-exponential function $\mathrm{E}_{\frac{1}{q}}(x)$, which is entire and has an infinite number of zeros for $x = -\frac{q^n}{1-q}$. The first q-exponential function is not suitable, since it is not entire and has an infine number of poles. This means that we can only define one q-Laplace transform.

Definition 31. Assume that $\mathrm{Re}(s) > \beta$, and let the function $f \in \Lambda_q$. Then the one-sided q-Laplace transform of f, as a function of s, is defined by

$$\mathcal{L}_q(s) \equiv \mathcal{L}_q(f(t)) \equiv \int_0^{\frac{1}{s(1-q)}} f(t) \mathrm{E}_{\frac{1}{q}}(-qst)\, d_q(t). \tag{14}$$

If f is discontinuous, we divide the q-integral into the corresponding continuous parts.

Formula (14) is perfectly well defined, since

1. The upper q-integral limit converges to ∞ for $\lim_{q\to 1^-}$
2. The value of the second q-exponential at the upper q-integral limit is the second zero of $\mathrm{E}_{\frac{1}{q}}$.
3. We chose the function $\mathrm{E}_{\frac{1}{q}}(-qst)$ to get simpler formulas for the q-Laplace transform, after q-integration by parts, which is often used in the proofs.

Remark 2. In the theory of the classical Laplace transform, functions $f(t)$ undergoing transformation are usually required to be 'of exponential type', so that the integral defining $F(s)$, which is over the interval $t \in (0,\infty)$, will converge for sufficiently large $\mathrm{Re}\, s$. The new q-Laplace transform involves a q-integral over a finite interval, from 0 to $\frac{1}{s(1-q)}$; so presumably the function $f(t)$ could fail to be of exponential type, without its q-Laplace transform failing to be defined; though its $\lim_{q\to 1^-}$ limit would not exist.

Remark 3. A q-Laplace transform $F(s)$ may exist without being defined for all s with $\mathrm{Re}\, s > 0$. Two examples are formulas (19) and (20).

The one-sided q-Laplace transform (14) has the following properties:
Linearity for q-Laplace transform:

$$\mathcal{L}_q(af(t)+bg(t)) = a\mathcal{L}_q(f(t)) + b\mathcal{L}_q(g(t)),\ f,g \in \Lambda_q. \tag{15}$$

Proof. This follows from the linearity of the q-integral. □

We shall compute some q-Laplace transforms, which occur most often.

Theorem 10. *The q-Laplace transform of a power function is given by*

$$\mathcal{L}_q(t^\alpha) = \frac{1}{s^{\alpha+1}}\Gamma_q(\alpha+1),\ \alpha \neq -n,\ n \in \mathbb{N}. \tag{16}$$

Proof. By q-integration by parts, we can show that the function

$$I(s,\alpha) \equiv \int_0^{\frac{1}{s(1-q)}} t^\alpha \mathrm{E}_{\frac{1}{q}}(-qst)\, d_q(t)$$

satisfies the recurrence

$$I(s,\alpha) = \frac{\{\alpha\}_q}{s} I(s,\alpha-1),$$

which is equivalent to (16). To this end put $u(t,q,\alpha) = t^\alpha$, $v(t,q,s) = -\frac{1}{s}\mathrm{E}_{\frac{1}{q}}(-st)$ in [8, (6.58)]. For $\mathrm{E}_{\frac{1}{q}}(-qst)$, use the q-derivative formula [8, (6.154)]. □

Corollary 11. *A q-analogue of* [20, (50), p. 163], [6, p. 192]. *The one-sided q-Laplace transform of a general q-hypergeometric series times a power function is given by*

$$\begin{aligned}&\mathcal{L}_q(t^{\lambda-1}\ {}_p\phi_{p-1}(a_1,\ldots,a_p;b_1,\ldots,b_{p-1}|q;t))\\ &\quad= \frac{\Gamma_q(\lambda)}{s^\lambda}\ {}_{p+1}\phi_p\left(\lambda,a_1,\ldots,a_p;b_1,\ldots,b_{p-1},\infty|q;(s(1-q))^{-1}\right).\end{aligned}$$

Corollary 12. *A q-analogue of* [20, (52), p. 164]*:*

$$\mathcal{L}_q(t^{\gamma-1}\ {}_2\phi_1(\alpha,\infty;\gamma|q;t(1-q))) = \frac{\Gamma_q(\gamma)}{s^\gamma}\frac{1}{(\frac{1}{s};q)_\alpha}.$$

Proof. Use the q-binomial theorem. □

Theorem 13. *Let* $\mathrm{H}(t-a)$ *denote the Heaviside function, where* $a \geq 0$*:*

$$\mathrm{H}(t-a) \equiv \begin{cases} 0, & 0 \leq t < a; \\ 1, & t \geq a. \end{cases}$$

The q-Laplace transform of $\mathrm{H}(t-a)$ *is given by*

$$\mathcal{L}_q(\mathrm{H}(t-a)) = \frac{1}{s}\mathrm{E}_{\frac{1}{q}}(-as). \tag{17}$$

Proof.

$$\int_a^{\frac{1}{s(1-q)}} \mathrm{E}_{\frac{1}{q}}(-qst)\,d_q(t) = \left[-\frac{1}{s}\mathrm{E}_{\frac{1}{q}}(-st)\right]_a^{\frac{1}{s(1-q)}}.$$

This equals the right hand side, since the upper limit is the first zero of $\mathrm{E}_{\frac{1}{q}}$. □

Theorem 14. *Let* $\delta(t-a)$ *denote the Dirac distribution, where* $a \geq 0$.
Then the q-Laplace transform of $\delta(t-t_0)$ *is given by*

$$\mathcal{L}_q(\delta(t-t_0)) = \widetilde{\mathrm{E}_{\frac{1}{q}}}(-st_0),\ t_0 \geq 0.$$

Proof. Put $\delta_a(t-t_0) \equiv \frac{1}{2a}\left[\mathrm{H}(t-(t_0-a)) - \mathrm{H}(t-(t_0+a))\right]$. The Dirac distribution can be expressed as

$$\delta(t-t_0) = \lim_{a\to 0}\delta_a(t-t_0).$$

By linearity and formula (17) this implies

$$\mathcal{L}_q(\delta_a(t-t_0)) = \frac{1}{2a}\left[\frac{1}{s}\left[\mathrm{E}_{\frac{1}{q}}(-s(t_0-a)) - \mathrm{E}_{\frac{1}{q}}(-s(t_0+a))\right]\right]. \tag{18}$$

This is an indeterminate expression 0/0 and we therefore use L'Hôpital's rule $\star$:

$$\mathcal{L}_q(\delta(t-t_0)) \overset{\text{by}(18)}{=} \lim_{a\to 0}\frac{1}{2s}\frac{\partial}{\partial a}\left[\sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}}{\{k\}_q!}\left[(as-st_0)^k - (-as-st_0)^k\right]\right]$$

$$\overset{\text{by}\star}{=} \lim_{a\to 0}\frac{1}{2}\sum_{k=1}^{\infty}\frac{k\cdot q^{\binom{k}{2}}}{\{k\}_q!}\left[\left[(as-st_0)^{k-1} + (-as-st_0)^{k-1}\right]\right] = \mathrm{RHS}. \quad \square$$

Lemma 15. *Scaling is given by*

$$\mathcal{L}_q(f(at)) = \frac{1}{a}\mathcal{L}_{q;a}f\left(\frac{s}{a}\right),\ a > 0.$$

Proof. Use formula (12). □

Theorem 16. *The q-Laplace transform of the q-exponential is given by*

$$\mathcal{L}_q(\mathrm{E}_q(\alpha t)) = \frac{1}{s-\alpha},\ \mathrm{Re}\left(\frac{\alpha}{s}\right) < 1. \tag{19}$$

Proof. In the end, the geometric series converges.

$$\int_0^{\frac{1}{s(1-q)}} \mathrm{E}_{\frac{1}{q}}(-qst)\mathrm{E}_q(\alpha t)\,d_q(t) = \sum_{n=0}^{\infty}\frac{\alpha^n}{\{n\}_q!}\int_0^{\frac{1}{s(1-q)}} \mathrm{E}_{\frac{1}{q}}(-qst)t^n\,d_q(t)$$
$$\overset{\text{by}(16)}{=} \sum_{n=0}^{\infty}\frac{\alpha^n}{\{n\}_q!}\frac{\Gamma_q(n+1)}{s^{n+1}} = \frac{1}{s}\sum_{n=0}^{\infty}\left(\frac{\alpha}{s}\right)^n = \frac{1}{s}\frac{1}{1-\frac{\alpha}{s}} = \mathrm{RHS}. \quad \square$$

Theorem 17. *The q-Laplace transform of the second q-exponential is given by*

$$\mathcal{L}_q\left(\mathrm{E}_{\frac{1}{q}}(\alpha t)\right) = \frac{1}{s}\ {}_1\phi_1\left(1;\infty|q;-\frac{\alpha}{s}\right),\ \mathrm{Re}\left(\frac{\alpha}{s}\right) < 1. \tag{20}$$

Proof. Similar to above, the series converges even better. □

Theorem 18. *The q-Laplace transform of q-Sine, where $a \in \mathbb{C}$ and $\mathrm{Re}\, s > \max[\mathrm{Re}(ia),\ \mathrm{Re}(-ia)]$ is given by*

$$\mathcal{L}_q(\mathrm{Sin}_q(\alpha t)) = \frac{\alpha}{s^2+\alpha^2}.$$

The q-Laplace transform of q-Cosine is given by

$$\mathcal{L}_q(\mathrm{Cos}_q(\alpha t)) = \frac{s}{s^2+\alpha^2}.$$

Proof. Use formulas (15) and (19). □

Theorem 19. *The q-Laplace transform of the nth iterated q-derivative can be expressed as a sum of $\mathrm{D}_q^i(f(0^+))$, the ith q-derivative of the function being transformed, evaluated at $t = 0$, or at least evaluated in the limit $t \to 0^+$:*

$$\mathcal{L}_q(\mathrm{D}_q^n(f(t))) = s^n\mathcal{L}_q(f(t)) - \sum_{i=0}^{n-1} s^{n-1-i}\mathrm{D}_q^i(f(0^+)),\ f \in \Lambda_q.$$

Proof. We use q-integration by parts [8, (6.59)].

$$\int_0^{\frac{1}{s(1-q)}} \mathrm{E}_{\frac{1}{q}}(-qst)\mathrm{D}_q^n(f(t))\,d_q(t) = [\mathrm{E}_{\frac{1}{q}}(-st)\mathrm{D}_q^{n-1}(f(t))]_0^\infty$$
$$+ s\int_0^{\frac{1}{s(1-q)}} q(-qst)\mathrm{D}_q^{n-1}(f(t))\,d_q(t)$$
$$= -\mathrm{D}_q^{n-1}(f(0)) + s[\mathrm{E}_{\frac{1}{q}}(-st)\mathrm{D}_q^{n-2}(f(t))]_0^\infty$$
$$+ s^2\int_0^{\frac{1}{s(1-q)}} \mathrm{E}_{\frac{1}{q}}(-qst)\mathrm{D}_q^{n-2}(f(t))\,d_q(t) + \cdots = \mathrm{RHS}. \qquad \square$$

Remark 4. Since $f \in \Lambda_q$, this is guaranteed to be finite, except possibly in the discontinuous points.

Corollary 20. *Initial and final value theorems.*

$$\lim_{s\to\infty} s\mathcal{L}_q(f) = \lim_{t\to 0^+} f(t),$$
$$\lim_{s\to 0} s\mathcal{L}_q(f) = \lim_{t\to\infty} f(t).$$

Proof. Use the previous theorem with $n = 1$. In the first case, the LHS goes to zero when $s \to \infty$. In the second case, cancel the two terms $f(0^+)$ on each side after letting $s \to 0^+$. On the LHS $\lim_{s\to 0^+} \frac{1}{s(1-q)} = +\infty$.

A simpler way is to use the formula (12) for multiplicative substitution in q-integral with the values $x = \frac{1}{s(1-q)}$ and $a = \frac{1}{s}$. $\square$

Theorem 21. *The q-Laplace transform of multiplication with a power function is given by*

$$\mathcal{L}_q(t^n f(t)) = (-1)^n q^{\binom{n}{2}} \mathrm{D}_{q,s}^n \mathcal{L}_q f(q^{-n}s),\ \ f \in \Lambda_q. \tag{21}$$

Proof.

$$(-1)^n q^{\binom{n}{2}} \mathrm{D}_{q,s}^n \mathcal{L}_q(q^{-n}s) = (-1)^n q^{\binom{n}{2}} \int_0^{\frac{1}{s(1-q)}} \mathrm{D}_{q,s}^n(\mathrm{E}_{\frac{1}{q}}(-q^{1-n}st))f(t)\,d_q(t)$$
$$\overset{\text{by[8, (6.154)]}}{=} \int_0^{\frac{1}{s(1-q)}} \mathrm{E}_{\frac{1}{q}}(-qst)t^n f(t)\,d_q(t) = \mathrm{LHS}. \qquad \square$$

Theorem 22. *The q-Laplace transform of multiplication with* $\mathrm{E}_q(\alpha t)$ *is given by*

$$\mathcal{L}_q(\mathrm{E}_q(\alpha t)f(t)) = \mathcal{L}_q f(s \uplus_q -\alpha),\ \ f \in \Lambda_q. \tag{22}$$

Proof.

$$\mathrm{LHS} \overset{\text{by(15)}}{=} \sum_{n=0}^{\infty} \frac{\alpha^n}{\{n\}_q!}\mathcal{L}_q(t^n f(t)) \overset{\text{by(21)}}{=} \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{\{n\}_q!} q^{\binom{n}{2}} \mathrm{D}_{q,s}^n \mathcal{L}_q f(q^{-n}s)$$
$$\overset{\text{by(8)}}{=} \mathrm{RHS}. \qquad \square$$

Example 1. We show a simple calculation.

$$\mathcal{L}_q(t\mathrm{Sin}_q(at)) \overset{\mathrm{by}(21)}{=} -\mathrm{D}_{q,s}\left[\frac{a}{\frac{s^2}{q^2}+a^2}\right] = \frac{\{2\}_q as}{(s^2+a^2)(s^2+q^2a^2)}.$$

The following table 1 shows the basic q-Laplace transforms.

$f(t)$	$\mathcal{L}_q(s)$
t^α	$\frac{1}{s^{\alpha+1}}\Gamma_q(\alpha+1)$
$t^{\gamma-1}\ {}_2\phi_1(\alpha,\infty;\gamma\vert q;t(1-q))$	$\frac{\Gamma_q(\gamma)}{s^\gamma}((\frac{1}{s};q)_\alpha)^{-1}$
$\mathrm{H}(t-a)$	$s^{-1}\mathrm{E}_{\frac{1}{q}}(-as)$
$\delta(t-t_0)$	$\widetilde{\mathrm{E}_{\frac{1}{q}}}(-st_0),\ t_0\geq 0$
$f(at)$	$\frac{1}{a}\mathcal{L}_{q;a}\left(\frac{s}{a}\right),\ a>0$
$\mathrm{E}_q(\alpha t)$	$\frac{1}{s-\alpha},\ \mathrm{Re}\left(\frac{\alpha}{s}\right)<1$
$\mathrm{E}_{\frac{1}{q}}(\alpha t)$	$\frac{1}{s}\ {}_1\phi_1\left(1;\infty\middle\vert q;-\frac{\alpha}{s}\right),\ \mathrm{Re}\left(\frac{\alpha}{s}\right)<1$
$\mathrm{Sin}_q(\alpha t)$	$\frac{\alpha}{s^2+\alpha^2}$
$\mathrm{Cos}_q(\alpha t)$	$\frac{s}{s^2+\alpha^2}$
$\mathrm{D}_q^n(f(t)$	$s^n\mathcal{L}_q(f(t))-\sum_{i=0}^{n-1}s^{n-1-i}\mathrm{D}_q^i(f(0))$
$t^nf(t)$	$(-1)^nq^{\binom{n}{2}}\mathrm{D}_{q,s}^n\mathcal{L}_qf(q^{-n}s)$
$\mathrm{E}_q(\alpha t)f(t)$	$\mathcal{L}_q(f(s\uplus_q-\alpha))$

TABLE 1. The q-Laplace transforms

4.3. Advanced q-hypergeometric transforms. We continue with some examples of q-Laplace transforms, which are all q-analogues of Exton [11, p. 223–224]. I moved Exton [11, A 6.1.15] and [11, A 6.1.16, p. 224] to the next theorem.

Theorem 23.

(23)
$$\frac{1}{\Gamma_q(a)}\mathcal{L}_q\left(t^{a-1}\ {}_2\phi_1\left[\begin{array}{c}2\infty\\ c\end{array}\middle|q;xt^2\right]\right) = \frac{1}{s^a}\ {}_4\phi_3\left[\begin{array}{c}\triangle(q;2;a)\\ c,2\infty\end{array}\middle|q;\frac{x}{(s(1-q))^2}\right].$$

(24)
$$\frac{1}{\Gamma_q(a)}\mathcal{L}_q\left(t^{a-1}\ {}_2\phi_1\left[\begin{array}{c}2\infty\\ a\end{array}\middle|q;xt\right]\right) = \frac{1}{s^a}\mathrm{E}_q\left(\frac{x}{s(1-q)^2}\right).$$

(25)
$$\frac{1}{\Gamma_q(a)}\mathcal{L}_q\left(t^{a-1}\ {}_2\phi_1\left[\begin{array}{c}2\infty\\ c\end{array}\middle|q;(x_1t\oplus_q x_2)\right]\right) = \frac{1}{s^a}\Upsilon_3\left(a;c\middle|q;\frac{x_1}{s(1-q)},x_2\right).$$

(26)
$$\frac{1}{\Gamma_q(a)}\mathcal{L}_q\left(t^{a-1}\ {}_2\phi_1\left[\begin{array}{c}b,\infty\\ c\end{array}\middle|q;(x_1t\oplus_q x_2)\right]\right) = \frac{1}{s^a}\Upsilon_1\left(b,a;c\middle|q;\frac{x_1}{s(1-q)},x_2\right).$$

(27)
$$\frac{1}{\Gamma_q(a)}\mathcal{L}_q\left(t^{a-1}\Psi_2\left(b;c,c'|q;x_1t,x_2\right)\right) = \frac{1}{s^a}\Psi_1\left(b,a;c,c'\middle|q;\frac{x_1}{s(1-q)},x_2\right).$$

(28)
$$\frac{1}{\Gamma_q(a)}\mathcal{L}_q\left(t^{a-1}\ {}_2\phi_1\left[\begin{array}{c}2\infty\\ c\end{array}\middle|q;x_1t\right]\ {}_2\phi_1\left[\begin{array}{c}2\infty\\ c'\end{array}\middle|q;x_2t\right]\right) = \frac{1}{s^a}\Psi_2\left(a;c,c'\middle|q;\frac{x_1}{s(1-q)},\frac{x_2}{s(1-q)}\right).$$

(29)
$$\frac{1}{\Gamma_q(a)}\mathcal{L}_q\left(t^{a-1}\ {}_2\phi_1\left[\begin{array}{c}2\infty\\ c\end{array}\middle|q;x_1t^2\right]\ {}_2\phi_1\left[\begin{array}{c}2\infty\\ c'\end{array}\middle|q;x_2t^2\right]\right) = \frac{1}{s^a}\Phi^{4:2}_{4:1}\left[\begin{array}{c}\triangle(q;2;a):2\infty;2\infty\\ 4\infty:c;c'\end{array}\middle|q;\frac{x_1}{(s(1-q))^2},\frac{x_2}{(s(1-q))^2}\right].$$

(30)
$$\frac{1}{\Gamma_q(a)}\mathcal{L}_q\left(t^{a-1}\Psi_2\left(b;c,c'|q;x_1t,x_2t\right)\right) = \frac{1}{s^a}\Phi_4\left(a,b;c,c'\middle|q;\frac{x_1}{s(1-q)},\frac{x_2}{s(1-q)}\right).$$

(31)
$$\frac{1}{\Gamma_q(a)}\mathcal{L}_q\left(t^{a-1}\ {}_2\phi_1\left[\begin{array}{c}b,\infty\\ c\end{array}\middle|q;x_1t\right]\ {}_2\phi_1\left[\begin{array}{c}b',\infty\\ c'\end{array}\middle|q;x_2t\right]\right) = \frac{1}{s^a}\Phi_2\left(a,b,b';c,c'\middle|q;\frac{x_1}{s(1-q)},\frac{x_2}{s(1-q)}\right).$$

$$
\begin{aligned}
&\frac{1}{\Gamma_q(a)}\mathcal{L}_q\left(t^{a-1}\Psi_2\left(b;a,c|q;x_1t,x_2\right)\right)\\
&\qquad =\frac{1}{s^a(\frac{x_1}{s(1-q)};q)_b}\ {}_3\phi_2\left[\begin{matrix} b,2\infty \\ c\end{matrix}\middle|q;x_2\middle\|\ \begin{matrix}\cdot\\ (\frac{x_1}{s(1-q)}q^b;q)_k\end{matrix}\right],\\
&\left|\frac{x_1}{s(1-q)}\right|<1.
\end{aligned}
\tag{32}
$$

Proof. First we prove (25). The left hand side equals

$$
\frac{1}{\Gamma_q(a)}\int_0^{\frac{1}{s(1-q)}} t^{a-1}\mathrm{E}_{\frac{1}{q}}(-qst)\sum_{m_1,m_2=0}^{\infty}\frac{(x_1t)^{m_2}x_2^{m_1-m_2}}{\langle c;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle 1;q\rangle_{m_1-m_2}}\,d_q(t)
$$

$$
\stackrel{\text{by(16)}}{=}\sum_{m_1,m_2=0}^{\infty}\Gamma_q\left[\begin{matrix}a+m_2\\ a\end{matrix}\right]\frac{x_1^{m_2}x_2^{m_1-m_2}}{\langle 1;q\rangle_{m_2}\langle 1;q\rangle_{m_1-m_2}\langle c;q\rangle_{m_1}s^{a+m_2}}\stackrel{\text{by[8, (1.46)]}}{=}\text{RHS}.
$$

Then we prove formula (32). The left hand side equals

$$
\frac{1}{\Gamma_q(a)}\int_0^{\frac{1}{s(1-q)}} t^{a-1}\mathrm{E}_{\frac{1}{q}}(-qst)\sum_{m_1,m_2=0}^{\infty}\frac{\langle b;q\rangle_{m_1+m_2}}{\langle a,1;q\rangle_{m_1}\langle c,1;q\rangle_{m_2}}(x_1t)^{m_1}x_2^{m_2}\,d_q(t)
$$

$$
\stackrel{\text{by(16)}}{=}\sum_{m_1,m_2=0}^{\infty}\Gamma_q\left[\begin{matrix}a+m_1\\ a\end{matrix}\right]\frac{\langle b;q\rangle_{m_1+m_2}}{\langle a,1;q\rangle_{m_1}\langle c,1;q\rangle_{m_2}s^{a+m_1}}x_1^{m_1}x_2^{m_2}
$$

$$
\begin{aligned}
&\stackrel{\text{by[8, (1.46)]}}{=}\frac{1}{s^a}\sum_{m_2=0}^{\infty}\frac{\langle b;q\rangle_{m_2}}{\langle c,1;q\rangle_{m_2}}x_2^{m_2}\\
&\qquad\times\sum_{m_1=0}^{\infty}\frac{\langle b+m_2\rangle_{m_1}}{\langle 1;q\rangle_{m_1}}\left(\frac{x_1}{s(1-q)}\right)^{m_1}\stackrel{\text{by[8, (7.27)]}}{=}\text{RHS}.
\end{aligned}
$$

□

The following formulas are all q-analogues of Erdélyi [7].

Theorem 24. *For* $\mathrm{Re}(b')>0,\ \mathrm{Re}(s)>0,\ \mathrm{Re}(s)>\mathrm{Re}(|y|)$, *we have a q-analogue of Erdélyi* [7, 1, p. 222]*:*

$$
\begin{aligned}
&\frac{1}{\Gamma_q(b')}\mathcal{L}_q\left(t^{b'-1}\ \Upsilon_1\left(a;b;c|q;x,yt\right)\right)\\
&\qquad=\frac{1}{s^{b'}}\ \Phi_1\left(a,b,b';c\middle|q;x,\frac{y}{s(1-q)}\right).
\end{aligned}
\tag{33}
$$

For $\mathrm{Re}(b)>0$, $\mathrm{Re}(s)>0$, $\mathrm{Re}(s)>\mathrm{Re}(|x|)$, *we have a q-analogue of Erdélyi* [7, 2, p. 222]*:*

$$
\begin{aligned}
&\frac{1}{\Gamma_q(b)}\mathcal{L}_q\left(t^{b-1}\ \Upsilon_2\left(a,a';c\middle|q;xt,y\right)\right)\\
&\qquad=\frac{1}{s^{b}}\ \Xi_1\left(a,a',b;c\middle|q;\frac{x}{s(1-q)},y\right).
\end{aligned}
\tag{34}
$$

For $\mathrm{Re}(c) > 0$, $\mathrm{Re}(s) > \max(\mathrm{Re}(|x|), \mathrm{Re}(|y|))$, *we have a q-analogue of Erdélyi* [7, 3, p. 222]*:*

$$\frac{1}{\Gamma_q(c)}\mathcal{L}_q\left(t^{c-1}\ \Upsilon_2\left(b, b'; c\middle|q; xt, y\right)\right) = \frac{1}{s^c\ \left(\frac{x}{(s(1-q))}; q\right)_b \left(\frac{y}{(s(1-q))}; q\right)_{b'}}. \tag{35}$$

For $\mathrm{Re}(a) > 0$, $\mathrm{Re}(s) > \max(\mathrm{Re}(|x|), \mathrm{Re}(|y|))$, *we have a q-analogue of Erdélyi* [7, 4, p. 222]*:*

$$\begin{aligned}&\frac{1}{\Gamma_q(a)}\mathcal{L}_q\left(t^{a-1}\ \Upsilon_2\left(b, b'; c\middle|q; xt, yt\right)\right)\\ &\quad= \frac{1}{s^a}\ \Phi_1\left(a, b, b'; c\middle|q; \frac{x}{s(1-q)}, \frac{y}{s(1-q)}\right).\end{aligned} \tag{36}$$

For $\mathrm{Re}(c) > 0$, $\mathrm{Re}(s) > \max(\mathrm{Re}(|x_i|), i = 1, \dots, n)$, *we have a q-analogue of Erdélyi* [7, 5, p. 222] *for the Humbert function:*

$$\frac{1}{\Gamma_q(c)}\mathcal{L}_q\left(t^{c-1}\ \Upsilon_2\left(\vec{b}; c\middle|q; \vec{xt}\right)\right) = \frac{1}{s^c\ \left(\frac{\vec{x}}{(s(1-q))}; q\right)_{\vec{b}}}. \tag{37}$$

For $\mathrm{Re}(a) > 0$, $\mathrm{Re}(s) > 0$, $\mathrm{Re}(s) > \mathrm{Re}(|x|)$, *we have a q-analogue of Erdélyi* [7, 6, p. 222]*:*

$$\frac{1}{\Gamma_q(a)}\mathcal{L}_q\left(t^{a-1}\ \Upsilon_3\left(b; c|q; tx, y\right)\right) = \frac{1}{s^a}\ \Xi_2\left(a, b; c\middle|q; \frac{x}{s(1-q)}, y\right). \tag{38}$$

For $\mathrm{Re}(b') > 0$, $\mathrm{Re}(s) > 0$, $\mathrm{Re}(s) > \mathrm{Re}(|y|)$, *we have a q-analogue of Erdélyi* [7, 7, p. 223]*:*

$$\frac{1}{\Gamma_q(b')}\mathcal{L}_q\left(t^{b'-1}\ \Upsilon_3\left(b; c|q; x, yt\right)\right) = \frac{1}{s^{b'}}\ \Upsilon_2\left(b, b'; c\middle|q; x, \frac{y}{s(1-q)}\right). \tag{39}$$

For $\mathrm{Re}(a) > 0$, $\mathrm{Re}(s) > 0$, $\mathrm{Re}(s) > \frac{\sqrt{y}}{1-q}$, *we have a q-analogue of Erdélyi* [7, 8, p. 223]*:*

$$\begin{aligned}&\frac{1}{\Gamma_q(2a)}\mathcal{L}_q\left(t^{2a-1}\ \Upsilon_3\left(b; c\middle|q; x, yt^2\right)\right)\\ &\quad= \frac{1}{s^{2a}}\ \Phi^{1:4;1}_{1:3;0}\left[\begin{array}{ccc} \infty: & \triangle(q; 2; 2a) & ; b \\ c: & 3\infty & ; \cdot \end{array}\middle| q; \frac{y}{(s(1-q))^2}, x\right].\end{aligned} \tag{40}$$

For $\mathrm{Re}(c) > 0$, $\mathrm{Re}(s) > 0$, $\mathrm{Re}(s) > \mathrm{Re}(|x|)$, *we have a q-analogue of Erdélyi* [7, 9, p. 223]*:*

$$\frac{1}{\Gamma_q(c)}\mathcal{L}_q\left(t^{c-1}\ \Upsilon_3(b; c\middle|q; xt, yt)\right) = \frac{1}{s^c\left(\frac{x}{(s(1-q))}; q\right)_b}\mathrm{E}_q\left(\frac{y}{s(1-q)^2}\right). \tag{41}$$

For $\mathrm{Re}(a) > 0$, $\mathrm{Re}(s) > 0$, $\mathrm{Re}(s) > \max(\mathrm{Re}(|x|), \mathrm{Re}(|y|))$, *we have a q-analogue of Erdélyi* [7, 10, p. 223]*:*

$$\frac{1}{\Gamma_q(a)}\mathcal{L}_q\left(t^{a-1}\ \Upsilon_3\left(b;c|q;xt,yt\right)\right) = \frac{1}{s^a}\ \Upsilon_1\left(a,b;c\middle|q;\frac{x}{s(1-q)},\frac{y}{s(1-q)}\right). \tag{42}$$

For $\mathrm{Re}(b') > 0$, $\mathrm{Re}(s) > 0$, $\mathrm{Re}(s) > \mathrm{Re}(|y|)$, *we have a q-analogue of Erdélyi* [7, 11, p. 223]*:*

$$\frac{1}{\Gamma_q(b')}\mathcal{L}_q\left(t^{b'-1}\ \Psi_1\left(a,b;c,c'\middle|q;x,yt\right)\right) = \frac{1}{s^{b'}}\ \Phi_2\left(a,b,b';c,c'\middle|q;x,\frac{y}{s(1-q)}\right). \tag{43}$$

For $\mathrm{Re}(b) > 0$, $\mathrm{Re}(s) > 0$, $\mathrm{Re}(s) > \mathrm{Re}(|x|)$, *we have a q-analogue of Erdélyi* [7, 12, p. 223]*:*

$$\frac{1}{\Gamma_q(b)}\mathcal{L}_q\left(t^{b-1}\ \Psi_2\left(a;c,c'\middle|q;x,yt\right)\right) = \frac{1}{s^{b'}}\ \Psi_1\left(a,b;c,c'\middle|q;\frac{x}{s(1-q)},y\right). \tag{44}$$

For $\mathrm{Re}(a) > 0$, $\mathrm{Re}(s) > \max(\mathrm{Re}(|x|), \mathrm{Re}(|y|))$, *we have a q-analogue of Erdélyi* [7, 13, p. 223]*:*

$$\frac{1}{\Gamma_q(a)}\mathcal{L}_q\left(t^{a-1}\ \Psi_2\left(b;c,c'\middle|q;xt,yt\right)\right) = \frac{1}{s^a}\ \Phi_4\left(a,b;c,c'\middle|q;\frac{x}{s(1-q)},\frac{y}{s(1-q)}\right). \tag{45}$$

For $\mathrm{Re}(b') > 0$, $\mathrm{Re}(s) > 0$, $\mathrm{Re}(s) > \mathrm{Re}(|y|)$, *we have a q-analogue of Erdélyi* [7, 14, p. 223]*:*

$$\frac{1}{\Gamma_q(b')}\mathcal{L}_q\left(t^{b'-1}\Xi_1\left(a,a';b;c\middle|q;x,yt\right)\right) = \frac{1}{s^{b'}}\ \Phi_3\left(a,a',b,b';c\middle|q;x,\frac{y}{s(1-q)}\right). \tag{46}$$

For $\mathrm{Re}(a') > 0$, $\mathrm{Re}(s) > 0$, $\mathrm{Re}(s) > \mathrm{Re}(|y|)$, *we have a q-analogue of Erdélyi* [7, 15, p. 223]*:*

$$\frac{1}{\Gamma_q(a')}\mathcal{L}_q\left(t^{a'-1}\Xi_2\left(a,b;c|q;x,yt\right)\right) = \frac{1}{s^{a'}}\Xi_1\left(a,a';b;c\middle|q;x,\frac{y}{s(1-q)}\right). \tag{47}$$

For $\mathrm{Re}(a') > 0$, $\mathrm{Re}(s) > 0$, $\mathrm{Re}(s) > 2\,\mathrm{Re}(\sqrt{y})$, *we have a* q*-analogue of Erdélyi* [7, 16, p. 223]*:*

$$\begin{aligned}&\frac{1}{\Gamma_q(2a')}\mathcal{L}_q\left(t^{2a'-1}\Xi_2\left(a,b;c\middle|q;x,yt^2\right)\right)\\&=\frac{1}{s^{2a'}}\Phi^{1:2;4}_{1:1;3}\left[\begin{array}{ccc}\infty: & a,b; & \triangle(q;2;2a')\\ c: & \infty; & 3\infty\end{array}\middle|q;x,\frac{y}{(s(1-q))^2}\right].\end{aligned}\tag{48}$$

Proof. First we prove (40). The left hand side equals

$$\frac{1}{\Gamma_q(2a)}\int_0^{\frac{1}{s(1-q)}}t^{2a-1}\mathrm{E}_{\frac{1}{q}}(-qst)\sum_{m_1,m_2=0}^{\infty}\frac{\langle b;q\rangle_{m_1}x^{m_1}(yt^2)^{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}}\,d_q(t)$$

$$\overset{\text{by}(16)}{=}\frac{1}{s^{2a}}\sum_{m_1,m_2=0}^{\infty}\Gamma_q\left[\begin{array}{c}2a+2m_2\\2a\end{array}\right]\frac{\langle b;q\rangle_{m_1}x^{m_1}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}}\left(\frac{y}{s^2}\right)^{m_2}$$

$$\overset{\text{by}[8,\,(1.46)]}{=}\frac{1}{s^{2a}}\sum_{m_1,m_2=0}^{\infty}\frac{\langle b;q\rangle_{m_1}\langle 2a;q\rangle_{2m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}}x^{m_1}\left(\frac{y}{(s(1-q))^2}\right)^{m_2}=\text{RHS}.$$

Then we prove (41). The left hand side equals

$$\frac{1}{\Gamma_q(c)}\int_0^{\frac{1}{s(1-q)}}t^{c-1}\mathrm{E}_{\frac{1}{q}}(-qst)\sum_{m_1,m_2=0}^{\infty}\frac{\langle b;q\rangle_{m_1}(xt)^{m_1}(yt)^{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}}\,d_q(t)$$

$$\overset{\text{by}(16)}{=}\frac{1}{s^{c}}\sum_{m_1,m_2=0}^{\infty}\Gamma_q\left[\begin{array}{c}c+m_1+m_2\\c\end{array}\right]\frac{\langle b;q\rangle_{m_1}x^{m_1}y^{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}s^{m_1+m_2}}$$

$$\begin{aligned}&\overset{\text{by}[8,\,(1.46)]}{=}\frac{1}{s^{c}}\sum_{m_2=0}^{\infty}\left(\frac{y}{s(1-q)}\right)^{m_2}\frac{1}{\langle 1;q\rangle_{m_2}}\\&\times\sum_{m_1=0}^{\infty}\left(\frac{x}{s(1-q)}\right)^{m_1}\frac{\langle b;q\rangle_{m_1}}{\langle 1;q\rangle_{m_1}}\overset{\text{by}[8,\,(7.27)]}{=}\text{RHS}.\end{aligned}$$

□

5. Conclusion

The table of q-Laplace transforms enables us to quickly find out which formula to be used. We are thus ready to solve inhomogenous q-difference equations, with right hand side for instance a delta function. The solutions will be the sum of the homogenous and the inhomogenous solutions like for differential equations. In the next paper we will solve the corresponding system of q-difference equations.

6. Discussion

It was not possible to find a q-analogue of the transform for $f(t-a)\mathrm{H}(t-a)$, since an additive substitution in q-integrals is not allowed. In the next step we could consider multiple q-Laplace transforms, which would then be q-analogues of well-known multiple Laplace transforms. The proof of the q-analogue of the Bromwich integral would require the corresponding Cauchy integral formula. We shall, however, try to discuss inversion problems in certain special cases in the next paper.

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