Two hierarchies of R-recursive functions

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Abstract

In the paper some aspects of complexity of R-recursive functions are considered. The limit hierarchy of R-recursive functions is introduced by the analogy to the μ-hierarchy. Then its properties and relations to the μ-hierarchy are analysed.

1. Introduction

The classical theory of computation deals with the functions on enumerable (especially natural) domains. The fundamental notion in this field is the notion of a (partial) recursive function. The problem of hierarchies for these functions is also in the interest of mathematicians (for elementary, primitive recursive function, Grzegorczyk hierarchy, compare [1].

During past years many mathematicians have been interested in creating analogous models of computation on real numbers (see for example Grzegorczyk [2], Blum, Shub, Smale [3]). An interesting approach was given by Moore. In the work [4] he defined a set of functions on the reals R (called R-recursive functions) in the analogous way to the classical recursive functions on the natural numbers N. His model has a continuous time of computation (a continuous integration instead of a discrete recursion). The great importance in Moore's model has the zero-finding operation μ, which is used to construct μ-hierarchy of R-recursive functions.

It was shown [5] that the zero-finding operator μ can be replaced by the operation of infinite limits. This allows us to define a limit hierarchy and relate it to μ-hierarchy.

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2. Preliminaries

We start with a fundamental definition of a class of real functions called **R**-recursive functions [4].

Definition 2.1 *The set of **R**-recursive functions is generated from the constants 0,1 by the operations:*

1) composition: $h(\overline{x}) = f\left(g(\overline{x})\right)$;

2) differential recursion: $h(\overline{x}, 0) = f(\overline{x}), \partial_y h(\overline{x}, y) = g(\overline{x}, y, h(\overline{x}, y))$ *(the equivalent formulation can be given by integrals:* $h(\overline{x}, y) = f(\overline{x}) + \int_0^y g(\overline{x}, y', h(\overline{x}, y')) dy'$ *);*

*3) **m**-recursion* $h(\overline{x}) = \mathbf{m}_y f(\overline{x}, y) = \inf\{y : f(\overline{x}, y) = 0\}$, *where infimum chooses the number **y** with the smallest absolute value and for two **y** with the same absolute value the negative one;*

4) vector-valued functions can be defined by defining their components.

Several comments are needed to the above definition. A solution of a differential equation need not be unique or can diverge. Hence, we assume that if **h** is defined by a differential recursion then **h** is defined only where a finite and unique solution exists. This is why the set of **R**-recursive functions includes also partial functions. We use (after [4]) the name of **R**-recursive functions in the article, however we should remember that in reality we have partiality here (partial **R**-recursive functions).

The second problem arises with the operation of infimum. Let us observe that if an infinite number of zeros accumulates just above some positive y or just below some negative y then the infimum operation returns that y even if it itself is not a zero.

In the papers [5, 6] it was shown that if in the Moore's definition [4] **m**-operation is replaced by infinite limits: $h(\overline{x}) = \liminf_{y \to \infty} g(\overline{x}, y)$, $h(\overline{x}) = \limsup_{y \to \infty} g(\overline{x}, y)$ then the resulting class of functions remains the same.

This gives us also the following result (including the limit operation in the form $h(\overline{x}) = \lim_{y \to \infty} g(\overline{x}, y)$, which can be in the obvious way obtained from limsup, liminf:

Corollary 2.2 *The class of **R**-recursive functions is closed under the operations of infinite limits:* $h(\overline{x}) = \liminf_{y \to \infty} g(\overline{x}, y)$, $h(\overline{x}) = \limsup_{y \to \infty} g(\overline{x}, y)$, $h(\overline{x}) = \lim_{y \to \infty} g(\overline{x}, y)$.

3. Hierarchies

The operator *m* is a key operator in generating the **R**-recursive functions. In a physical sense it has a property of being strongly uncomputable. This fact suggests creating a hierarchy, which is built with respect to the number of uses of *m* in the definition of a given *f*.

Definition 3.1 *([4]) For a given **R**-recursive expression* $s(\overline{x})$, *let* $M_{x_i}(s)$ *(the **m**-number with respect to* x_i *) be defined as follows:*

$$M_x(0) = M_x(1) = M_x(-1) = 0, \tag{1}$$

$$M_x\big(f(g_1, g_2, ...)\big) = \max_j \big(M_{x_j}(f) + M_x(g_j)\big), \tag{2}$$

$$M_x\left(h = f + \int_0^y g(\overline{x}, y', h)\, dy'\right) = \max\big(M_x(f), M_x(g), M_h(g)\big), \tag{3}$$

$$M_y\left(h = f + \int_0^y g(\overline{x}, y', h)\, dy'\right) = \max\big(M_{y'}(g), M_h(g)\big), \tag{4}$$

$$M_x\big(m_y f(\overline{x}, y)\big) = \max\big(M_x(f), M_y(f)\big) + 1, \tag{5}$$

where x *can be any* $x_1, ..., x_n$ *for* $\overline{x} = (x_1, ..., x_n)$.

For an **R**-recursive function *f*, let $M(f) = \max_{x_i}(s)$ minimized over all expressions *s* that define *f*. Now we are ready to define M-hierarchy (*m*-hierachy) as a family of $M_j = \{f : M'(f) \le j\}$.

Let us construct the analogous definition of L-hierarchy by replacing in the above definition M_x by L_x and changing line (5) to the following form (5'):

$$L_x\left(\liminf_{y \to \infty} g(\overline{x}, y)\right) = L_x\left(\limsup_{y \to \infty} g(\overline{x}, y)\right) =$$

$$= L_x\left(\lim_{y \to \infty} g(\overline{x}, y)\right) = \max\big(L_x(f), L_y(f)\big) + 1.$$

For an **R**-recursive function *f*, let $L(f) = \max_i L_{x_i}(s)$ minimized over all expressions *s* that define *f* without using the *m*-operation.

Definition 3.2 *The **L**-hierarchy is a family of* $L_j = \{f : L(f) \le j\}$.

Let us add that in Definition 3.2 we use explicitly the operator $f(\overline{x}) = \lim_{y \to \infty} g(\overline{x}, y)$ to avoid its construction by other operators (lim sup, lim inf), which would efect in a superficially higher class of a complexity of a function f.

As an obvious corollary from definitions we have the following statement.

Lemma 3.3 *The classes* M_0 *and* M_1 *are identical.*

A function $f \in L_0 = M_0$ will be called (by an analogy to the case of natural recursive functions) a primitive **R**-recursive function. After Moore [4] we can conclude that such functions as: $-x$, $x + y$, xy, x/y, e^x, $\ln x$, y^x, $\sin x$, $\cos x$ are primitive **R**-recursive.

We can give a few results on some levels of the limit hierarchy.

Lemma 3.4. *The Kronecker **d** function, the signum function and absolute value belong to the first level (* L_1 *)of limit hierarchy.*

Proof. It is sufficient to take the following definitions [5]: hence $d(0) = 1$ and for all $x \ne 0$ we have $d(x) = 0$ let us define $d(x) = \liminf_{y \to \infty} \left(\dfrac{1}{1+x^2} \right)^y$. Now from the expression $\liminf_{y \to \infty} \arctan xy = \begin{cases} p/2, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -p/2, & \text{if } x < 0, \end{cases}$ we obtain

$$\text{sgn}(x) = \frac{\liminf_{y \to \infty} \arctan xy}{2 \arctan 1} \text{ and } |x| = \text{sgn}(x) x.$$

We should be careful with definions of functions by cases:

Lemma 3.5 *For* $h(\overline{x}) = \begin{cases} g_1(\overline{x}), & \text{if } f(\overline{x}) = 0, \\ g_2(\overline{x}), & \text{if } f(\overline{x}) = 1, \\ \mathbf{M} & \mathbf{M} \\ g_k(\overline{x}), & \text{if } f(\overline{x}) \ge k-1 \end{cases}$ *and* $g_i \in L_{n_i}$ *for all* $1 \le i \le k$,

$f \in L_m$ *the function* h *belongs to* $L_{\max(n_1, \dots n_k, m+1)}$

Proof. Let us see that $eq(x,y)=d(x-y)\in L_1$ and

$$ge(x,y)=\frac{\left(\text{sgn}(x-y)+eq(x,y)\right)}{2}+\frac{1}{2}\in L_1.$$ Then of course

$$h(\overline{x})=\sum_{i=1}^{k-1}g_i(\overline{x})eq\left(f(\overline{x}),i-1\right)+g_k(\overline{x})ge\left(f(\overline{x}),k-1\right).\square$$

Of course this result can be easily extended to other forms of definitions by cases.

Lemma 3.6 *The function $\Theta(x)$ (equal to 1 if $x\geq 0$, otherwise 0), maximum $\max(x,y)$, square-wave function s are in L_2, the function $p(x)$ such that $p(x)=1$ for $x\in[2n,2n+1]$ and $p(x)=0$ for $x\in[2n+1,2n+2]$ is in L_2 and the floor function $\lfloor x\rfloor$ is in L_3.*

Proof. We give the proper definitions (from [6]) for these functions. Let

$$\Theta(x)=d\left(x-|x|\right),$$

$$\max(x,y)=xd(x-y)+\left(1-d(x-y)\right)\left[x\Theta(x-y)+y\Theta(y-x)\right],$$

$$s(x)=\Theta\left(\sin(px)\right).$$

The function $p(x)$ can be given as $s(x)\left(1-d\left(\sin\frac{(x-1)p}{2}\right)\right)$, so $p\in L_2$.

The floor function we can define by the auxiliary function $w(0)=0$, $\partial_x w(x)=2\Theta\left(-\sin(2px)\right)$ as

$$\lfloor x\rfloor=\begin{cases}2w(x/2) & \text{if }p(x)=1,\\ 2w\left((x-1)/2\right) & \text{if }p(x)=0.\end{cases}$$

From the above equation we have $\lfloor x\rfloor$ in L_3. $\square$

Let us recall that if $f:R^n\to R$ is an **R**-recursive function then the function $f_{iter}(i,\overline{x})$ is **R**-recursive, too.

Lemma 3.7 *Let $f:R^n\to R$ belongs to the class L_i, then we have $f_{iter}:R^{n+1}\to R$ is in $L_{\max(2,j)}$.*

Proof. The definitions, which were given by Moore [3] $f_{iter}(i,\overline{x})=h(2i)$, where

$$h(0)=g(0)=\overline{x},$$

$$\partial_t g(t) = \big[f\big(h(t)\big) - h(t) \big] s(t),$$

$$\partial_t h(t) => \left[\frac{g(t) - h(t)}{r(t)} \right] \big(1 - s(t)\big),$$

with s - a square wave function in L_2 and $r(0) = 0$, $\partial_t r(t) = 2s(t) - 1$, $r, s \in L_2$ give us the desirable statement. $\square$

Lemma 3.8 *The $\mathbf{R}^l$-recursive functions $g_2 : R^2 \to R$, $g_2^1, g_2^2 : R \to R$ such that $(\forall x, y \in R) g_2^1 \big(g_2(x,y) \big) = x$, $(\forall x, y \in R) g_2^2 \big(g_2(x,y) \big) = y$, have the following properties: g_2, g_2^1 are in L_{10}, g_2^2 is in L_{14}.*

Proof. We have the auxiliary functions Γ_2, Γ_2^1, Γ_2^2, which are coding and decoding functions in the interval $(0,1)$: $\Gamma_2(x,y) = c(x) + c(y)/10$, where

$$c(x) = \lim_{i \to \infty} z\big(a(i,x)\big) / 10^{2i} + b(i,x)/10^i ,$$

and later $z(x) = \lim_{i \to \infty} z'_{iter}(i,x)$,

$$z'_{iter}(i, a_1...a_n.a_{n+1}...) = a_1...a_n 0...a_{n+1} 0.a_{n+i+1}... ,$$
$$a(i, 0.a_1 a_2 ... a_i ...) = 0.a_1...a_i$$
$$b(i, 0.a_1 a_2 ... a_i ...) = 0.\underbrace{0...0}_{i} a_{i+1}... ,$$

$$(z'(x) = \begin{cases} 100\lfloor x \rfloor + 10\big(x - \lfloor x \rfloor\big), & \text{if } \lfloor x \rfloor \neq x, \\ x, & \text{if } \lfloor x \rfloor = x; \end{cases} \in L_4, a, b \in L_4. \text{ Also } z'_{iter} \text{ belongs}$$

to L_4, hence $\Gamma_2(x,y) \in L_{10}$, decoding of the first element is described in the symmetric way so $\Gamma_2^1(x)$ is in L_{10}, but $\Gamma_2^2(x) = \Gamma_2^1\big(10 - \lfloor 10x \rfloor\big)$ so $\Gamma_2^2 \in L_{14}$.

The functions Γ_2, Γ_2^1, Γ_2^2 can be extended to all reals by one-to-one $f : (0,1) \to R \in L_0$ without the loss of their class. $\square$

The same method of coding and decoding by interlacing of ciphers (only the power of 10 should be changed) gives us the functions $g_n : R^n \to R$ and $g_n^i : R \to R$ for $i = 1,...,n$ such that

$$(\forall i)(\forall x_1 ..., x_n \in R) g_n^i \big(g_n(x_1,...,x_n) \big) = x_i$$

in the same class: $g_n, g_n^1 \in L_{10}$ and $(\forall i > 1) g_n^i \in L_{14}$.

We finish this part with the important form of defining: a new function is given as a product of values f in some integer points.

Lemma 3.9 *There exists such constant* $p \in N$ *that for the function*

$$\prod_{z=0}^{y} f(\overline{x}, z) = \begin{cases} f(\overline{x},0) f(\overline{x},1)...f(\overline{x},\lfloor y-1 \rfloor), & \text{if } y \geq 1, \\ 1, & \text{if } 0 \leq y < 1, \\ 0, & \text{if } y < 0, \end{cases}$$

if the function f is in the class L_m *then* $\prod_{z=0}^{y} f(\overline{x}, z)$ *is in the class* L_{m+p} *(p is independent of m).*

Proof. By the definitions

$$t(w) = g_{n+2}\left(g_{n+2}^{1,n}(w), g_{n+2}^{n+1}(w)+1, f\left(g_{n+2}^{1,n}(w), g_{n+2}^{n+1}(w)\right) \cdot g_{n+2}^{n+2}(w)\right)$$

and

$$S(\overline{x}, z) = \underbrace{t(...t}_{\lfloor z \rfloor}(s(\overline{x},0))...) = t_{iter}\left(\lfloor z \rfloor, g_{n+2}(\overline{x},0,1)\right)$$

we get the property

$$\prod_{y=0}^{z} f(\overline{x}, y) = g_{n+2}^{n+2}\left(S(\overline{x}, z)\right).$$

From the defintion of the limit hierarchy we get $\prod_{y=0}^{z} f(\overline{x}, y) \in L_{m+38}$. $\square$

In the rest of the paper we will use the constant *p* as the number of limits used in the recursive defintion of the product $\prod_{y=0}^{z} f(\overline{x}, y)$ instead of the value 38. The above constructions are tedious and can be improved with a better approximation of *p*.

4. Main results

Now we are ready to formulate two theorems which demonstrate connections between L-hierarchy and M-hierarchy.

Theorem 4.1 *Let* $f : R^n \rightarrow R$ *be an* **R***-recursive function. Then if* $f \in L_i$ *then* $f \in M_{10i}$.

Proof. We use a simple induction here. The case $i = 0$ is given in Lemma 3.3. Now let us suppose that the thesis is true for $i = n$. Let $f \in L_{n+1}$ be defined as $f(\overline{x}) = \lim_{y \to \infty} g(\overline{x}, y)$ for $g \in L_n$. Then we can recall Theorem 4.2 from [6] which gives us the following result: to define **f** from *g* it is necessary to use at

most 10 **m**-operation. Hence for $g \in M_{10n}$ the function **f** satisfies $f \in M_{10n+10}$. Similar inferences hold for $\lim\inf, \lim\sup.$ □

Now we can give the result about the 'limit complexity' of the infimum operator **m**.

Lemma 4.2 *If* $f(\overline{x}, y): R^{n+1} \to R$ *is in the class* L_m *then the function* $g: R^n \to R$, $g(\overline{x}) = m_y f(\overline{x}, y)$ *is in the class* L_{m+3p+9} *is from Lemma 3.9.*

Proof. Here we must employ the results from [6]. There we defined the function $g: R^n \to R$, $g(\overline{x}) = m_y f(\overline{x}, y)$ for $f(\overline{x}, y): R^{n+1} \to R$ (f - **R**-recursive) replacing the **m**-operator by limit operation. First we introduced the function

$$Z^f(\overline{x}, z) = \begin{cases} \inf_y \{ f : K^f(\overline{x}, y) = 0 \}, & \text{if } z = 0 \text{ and } \exists y K^f(\overline{x}, y) = 0, \\ \text{undefined} & \text{if } z = 0 \text{ and } \forall y K^f(\overline{x}, y) \neq 0, \\ 1 & \text{if } z \neq 0, \end{cases}$$

given in the following way:

$$Z^f(\overline{x}, z) = \begin{cases} \text{undefined} & \text{if } (z = 0) \wedge \left(S^f(\overline{x}) < \frac{1}{12} \right), \\ \sqrt{S^f(\overline{x}) - \frac{1}{12}}, & \text{if } (z = 0) \wedge \left(S^f(\overline{x}) \geq \frac{1}{12} \right) \\ & \wedge f\left(\overline{x}, \sqrt{S^f(\overline{x}) - \frac{1}{12}} \right) = 0, \\ -\sqrt{S^f(\overline{x}) - \frac{1}{12}}, & \text{if } (z = 0) \wedge \left(S^f(\overline{x}) \geq \frac{1}{12} \right) \\ & \wedge f\left(\overline{x}, -\sqrt{S^f(\overline{x}) - \frac{1}{12}} \right) = 0, \\ 1, & \text{if } z \neq 0. \end{cases}$$

where $S^f(\overline{x}) = \lim_{t \to \infty} S_1^f(\overline{x}, t) + \lim_{t \to \infty} S_2^f(\overline{x}, t)$. Both functions S_1^f, S_2^f are defined by an integration

$$S_i^f(\overline{x}, t) = \int y^2 \left(1 - h^f\left(\overline{x}, (-1)^{i+1} y - 1/2, (-1)^{i+1} y + 1/2 \right) \right) dy, \ i = 1, 2$$

from $h^f(\overline{x}, a, b) = \liminf_{t \to \infty} \prod_{w=0}^{z+1} K^f\left(\overline{x}, a + w \frac{b-a}{z} \right)$ where K^f is the characteristic function of f.

Hence we can conclude that if K^f is in the L_s then Z_f is in the class L_{s+p+3}. Let us finish with the definition of the characteristic function of the infimum of zeros of f (see Theorem 4.2 from [5]

$$K_m^f(y) = 1 - \lim_{a \to -\infty} \lim_{b \to \infty} \lim_{z \to \infty} G^f(\overline{x}, z, a, b, y),$$

where $G^f(\overline{x}, z, a, b, y)$ divides the interval $[a,b]$ into $2^{\lfloor z \rfloor}$ equal subintervals and gives the value 1 for y from the subintervals, which contains the least zero of f in $[a,b]$ and value 0 otherwise. Precisely for y from $\left[a, a + \dfrac{b-a}{2^{\lfloor z \rfloor}}\right]$

$$G^f(\overline{x}, z, a, b, y) = \begin{cases} 1, & \text{if } h^f\left(\overline{x}, a, a + \dfrac{b-a}{2^{\lfloor z \rfloor}}\right) = 0, \\ 0, & \text{otherwise} \end{cases}$$

for $y \in \left(a + \dfrac{(k-1)(b-a)}{2^{\lfloor z \rfloor}}, a + \dfrac{k(b-a)}{2^{\lfloor z \rfloor}}\right)$ (where $k = 2, 3, ..., 2^n$) we have:

$$G^f(\overline{x}, z, a, b, y) = \begin{cases} 1, & \text{if } \prod_{i=1}^{k-1} h^f\left(\overline{x}, a + \dfrac{(i-1)(b-a)}{2^{\lfloor z \rfloor}}, a + \dfrac{i(b-a)}{2^{\lfloor z \rfloor}}\right) \neq 0 \\ & \wedge h^f\left(\overline{x}, a + \dfrac{(k-1)(b-a)}{2^{\lfloor z \rfloor}}, a + \dfrac{k(b-a)}{2^{\lfloor z \rfloor}}\right) = 0, \\ 0, & \text{otherwise} \end{cases}$$

and for $Y \notin [A, B]$ the function g_x^f is equal to 2.

The definition of G_f is given by the cases with respect to the value of the expression given by $\prod h^f$, since for $f \in L_m$, the function $h_f \in L_{m+p+2}$ and $G^f \in L_{m+2p+3}$. Then we have $K_m^f \in L_{m+2p+6}$. Now we must use the function K_m^f in the same way as K^f which gives us Z_f in the class L_{m+3p+9}. The final definition of $g(\overline{x}) = m_y f(\overline{x}, y)$ ([5] Theorem 4.3) given below

$$g(\overline{x}) = \begin{cases} Z^{f^+}(\overline{x},0) - Z^{f^-}(\overline{x},0), & \text{if } S^{f^+}(\overline{x}) < \frac{1}{12} \wedge S^{f^-}(\overline{x}) < \frac{1}{12}, \\[2ex] Z^{f^+}(\overline{x},0), & \text{if } \left(S^{f^+}(\overline{x}) \geq \frac{1}{12} \wedge S^{f^-}(\overline{x}) < \frac{1}{12} \right) \\ & \text{or} \\ & \left(S^{f^+}(\overline{x}) < \frac{1}{12} \wedge S^{f^-}(\overline{x}) < \frac{1}{12} \right. \\ & \left. \wedge Z^{f^+}(\overline{x},0) < Z^{f^-}(\overline{x},0) \right), \\[2ex] -Z^{f^-}(\overline{x},0), & \text{if } \left(S^{f^+}(\overline{x}) < \frac{1}{12} \wedge S^{f^-}(\overline{x}) \geq \frac{1}{12} \right) \\ & \text{or} \\ & \left(S^{f^+}(\overline{x}) < \frac{1}{12} \wedge S^{f^-}(\overline{x}) < \frac{1}{12} \right. \\ & \left. \wedge Z^{f^+}(\overline{x},0) \geq Z^{f^-}(\overline{x},0) \right), \end{cases}$$

where $f^+(\overline{x},y) = \begin{cases} f(\overline{x},y), & y \geq 0, \\ 1, & y < 0; \end{cases}$ $f^-(\overline{x},y) = \begin{cases} f(\overline{x},-y), & y > 0, \\ 1, & y \leq 0; \end{cases}$ remains the

class of g identical to the class of Z^f, i.e. $g \in L_{m+3p+9}$. $\square$

Theorem 4.3 *Let* $f : R^n \to R$ *be an* **R***-recursive function. Then for all* $i \geq 0$ *if* $f \in M_i$ *then* $f \in L_{(3p+9)i}$.

The above statement is a simple consequence of the fact $M_0 = L_0$ and Lemma 4.2.

5. Conclusions

In the paper we give the first rough approximation of 'a complexity' of limit operations in the terms of the *m*-operator and conversely. The results, interpreted in the intuitional way, can suggest what kind of connection exists between infinite limits and a *m*-operator.

We also establish the proper relation between the levels of the limit hierarchy and *m*-hierarchy. Let us point out that in consequence we may investigate analogies which exist for the limit hierarchy (also *m*-hierarchy) and Baire classes

[7]. Also the kind of a connection between the $\sum_{n}^{0}$ – measurable functions and R-recursive functions is an open problem.

Acknowledgments

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